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MATHEMATICAL COMPUTER TOOLS IN PREPARING OF PROFESSIONAL MATHEMATICIANS

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What are Computer Tools for Mathematicians?

The computer tools for mathematicians are:

- Tools for symbolic calculations
- Tools for mechanical theorem proving
- Tools for learning of mathematics
- Tools for teaching of mathematics
- Tools for programming
- Tools for visualization
- Tools for preparing publications

But tools in mathematics are not only tools!

1. Computer Tools Give Us a New Glance on How to Work with Basic Mathematical Concepts

Two simple examples are:

- Differentiation
- Integration

Differentiation as in Calculus and as in Computer Tools

$$y = \log_x \left(x^{\sin(x)} + 1 \right)$$

Why when studying calculus some students has problem with algorithm of differentiation of such a function?

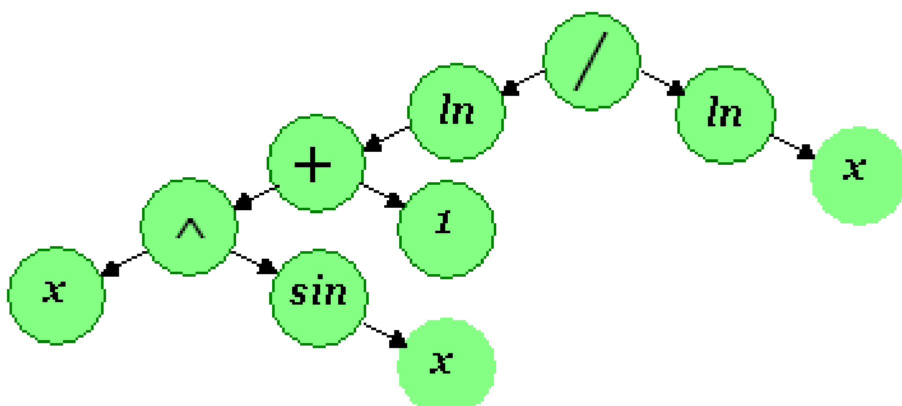
- Because most of the problems are behind the algorithm!
- Student must translate (or interpret) such a strange notions where one "x" is beneath the row and other is above the row!
- After that he must transform this formula to such a form when standard rules begin to work.
- After differentiation he must transform result to "nice" form.

How Computer Tool Begin to do Differentiation?

First of all one must input formula in computer and it will be impossible without using other form of representation.

$$\frac{\ln \left(x^{\sin(x)} + 1 \right)}{\ln(x)}$$

Next step do computer and find simple internal form for this formula. From human point of view it is a syntax tree.



How Computer Tool Complete Differentiation?

Afterward we use simple recursive algorithm which is another form of basic differential rules which transform tree of a function to the tree of its derivative.

After that we must try others not so evident algorithms to simplify the tree.

So one can see that such an analysis give us a new glance on differentiation technique.

Integration

- During the long period humans thought integration to be a set of heuristics (for example famous French mathematician Jean Dieudonne gave an advice to replace all geometric school problems with analytic problems of how to find primitives).
- But about twenty years ago it was proved that there exist algorithm which find primitives of all functions which are a composition of elementary ones (or say about impossibility to give such a representation).
- It is important to note that this algorithm is a generalization of algorithm for rational functions.

So it become important not to practice on searching of primitives but to explore abstract algebra.

The First Conclusion

The knowledge of how mathematical computer tools are working can help a student

- To separate abstract ideas from concrete interpretation
- To become free in using computer tools and sure in reliability of results

Remember that "tools in mathematics are not only tools!", so

2. Computer Tools Give a New Glance on How to Get New Results

A first example of course is Four Color Map Problem which was solved with essential using of computer tools. There was necessary large enumerations which were impossible for humans.

A division of a work between a computer and a human become a fresh idea which attracts young people to mathematics.

An example of that is ACM Programming Contest for Universities. Most of programmers said that it is Mathematics Contest because most of the problems are mathematical ones.

Example: Two ACM Contest Problems

Bicoloring: In 1976 the "Four Color Map Theorem" was proven with the assistance of a computer. This theorem states that every map can be colored using only four colors, in such a way that no region is colored using the same color as a neighbor region. Here you are asked to solve a simpler similar problem. You have to decide whether a given arbitrary connected graph can be bicolored

Carmichael Numbers: One of them is the Fermat test. Let a be a random number between 2 and $n-1$ (being n the number whose primality we are testing). Then, n is probably prime if the following equation holds:

$$a^n \bmod n = a$$

If a number passes the Fermat test several times then it is prime with a high probability. Some numbers that are not prime still pass the Fermat test with every number smaller than themselves. These numbers are called Carmichael numbers. In this problem you are asked to write a program to test if a given number is a Carmichael number.

The Second Conclusion

The enumeration can be a part of a mathematical result.

We must support activity in translation of some traditional mathematical problems into enumeration ones when using computer tools and to expand types of such a problems because such an activity helps student to understand complexity of mathematical problem, "enter" it.

3. Computer Tools Provide Mathematicians with an "Intellectual" Support for Mathematical Research

It was mechanics (which is part of classical mathematics) where the first try to mechanize process of new results receiving was done.

Example: French astronomer S. Delone in about 1860 gave formulas for Moon moving which took into account many factors. He calculated the rows and gave solution in the form of the row for small parameter to 10th degree of it.

But he made a mistake in 8th degree which was found by computer algebra system only in 1966.

It is important that mechanics tried to construct computer systems solving equations analytically instead of doing such a work "by hands".

Computer Algebra Tools

Now every student has an access to computer algebra tools which are embedded in such popular systems as Maple or Mathematica.

But scientific work imply development of such systems for more specialized problems.

Examples: Special mechanical application include producing of equations for dynamical systems. Symbol analysis can be used to get conjugate system equations in optimization problems, to generate equations in variations for solution of boundary problems, to search stationary solutions and so on.

The Third Conclusion

Mathematicians which plan to work in applied mathematics must be prepared in three directions:

- They must be prepared for free using of such a system as Maple or similar.
- They must be involved in constructing of new systems of computer algebra or to realization of mathematical algorithms.
- They must participate in solution of mathematical problems in applied mathematics by using of systems for symbol processing.

What about "pure" mathematicians?

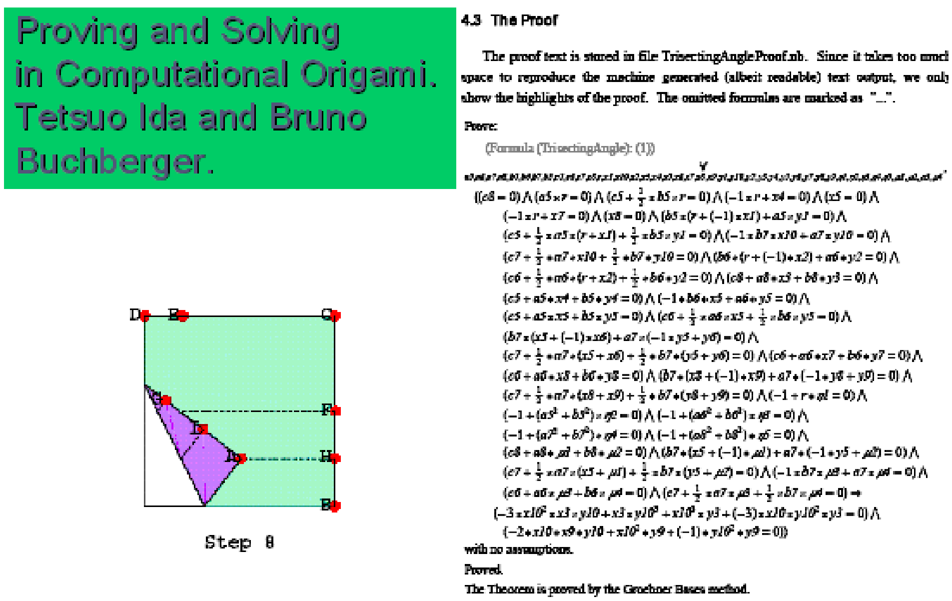
4. Computer Tools Give a New Glance on Representation of Abstract Concepts and Manipulation with Them for Research and Teaching

Example: About twenty years ago Shang-Ching Chou by using Methods in Mechanical Geometry Theorem Proving (Wus Method and The Grobner Basis Method) creates program which can easily prove one theorem from classical geometry after another and occasionally answer an open question.

Works mentioned above gave a new direction for Integration of Symbolic Computation and Mechanized Reasoning.

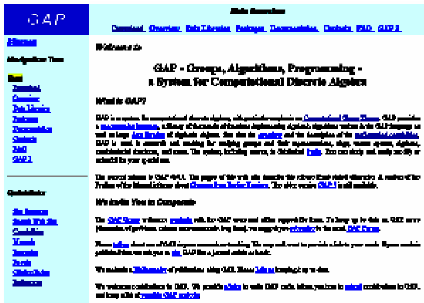
So it became interesting for pure mathematicians to try to use same methods (symbolic processing) for elimination of some routine phases in proving process.

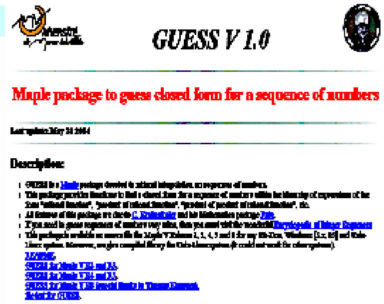
EXAMPLE [5]**:**



Activity in Other Directions of Pure Mathematics: Two Examples

- GAP - Groups, Algorithms, Programming – a System for Computational Discrete Algebra. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects.
- GUESS is a Maple package devoted to rational interpolation on sequences of numbers. This package provides functions to find a closed form for a sequence of numbers within the hierarchy of expressions of the form "rational function", "product of rational function", "product of product of rational function", etc.

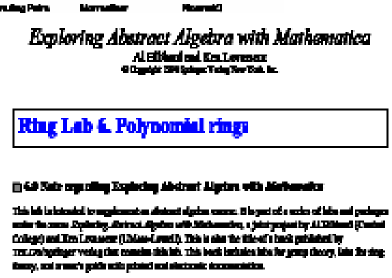




Computer Tools in Teaching of Pure Mathematics

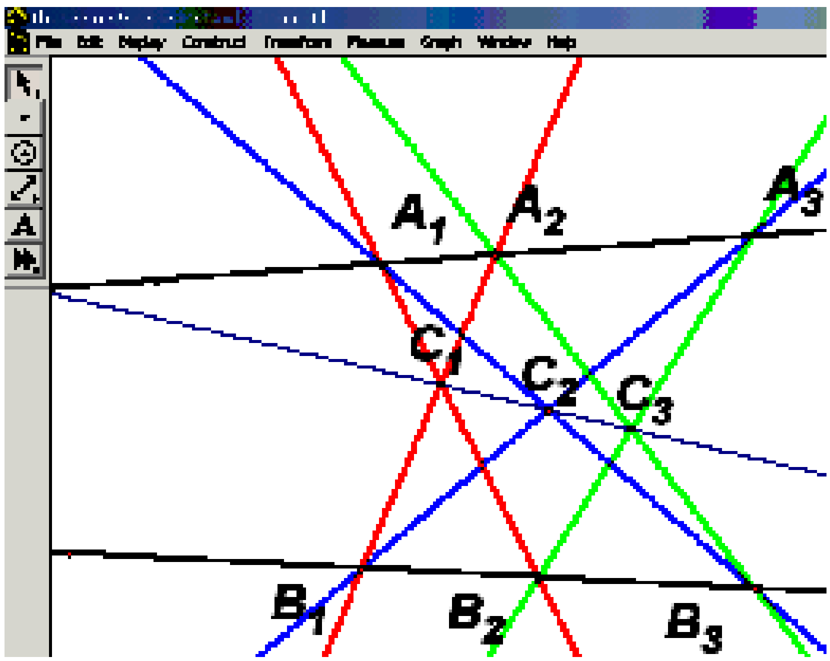
A possibility to manipulate with an abstract concepts as with real objects gives the new way to organize practice in algebra.

Example: Abstract Algebra with Mathematica, is intended for anyone trying to learn (or teach) abstract algebra (a difficult course for some students). The Mathematica labs are designed to appeal to visualization of various algebraic ideas (as pioneered by Ladnor Geissinger in his software package Exploring Small Groups). Additionally, the nature of the Mathematica notebooks encourages an exploratory environment in which one can make and test conjectures.



Computer Tools in Teaching of School Mathematics

- Example: Dynamic Geometry Tools.
- Experimental checking of conjectures and underlying mathematics:
  - Randomized Proving of Pappo's theorem.
  - "We will prove Pappo's theorem using algorithm ... First of all we have to find a construction sequence that encodes Pappo's theorem as a constructive incidence theorem.
  - Next we only have to evaluate 4^{15} (roughly a billion) examples of Pappo's theorem and when we do not find a counter-example this will be a proof of the theorem".



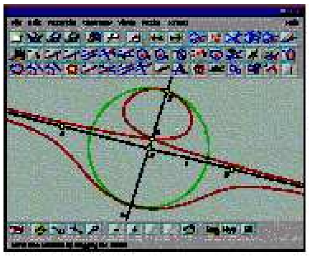
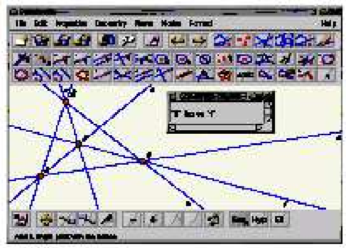
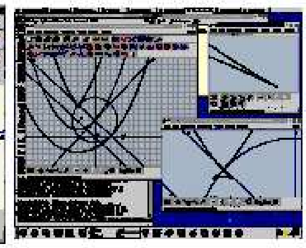
GSP JSP

The Fourth Conclusion

Mathematical computer tools give us a way to do some changes in tutorials and to organize practice in pure mathematics.

Other function of computer instrument – to be boundary object in storing and transferring of mathematical ideas in algorithmic form.

5. Construction of Computer Tools Leads to a New Field of Mathematical Research: The Fifth Conclusion

Example: Prof. Dr. Ulrich Kortenkamp, Technische Universitt Berlin:

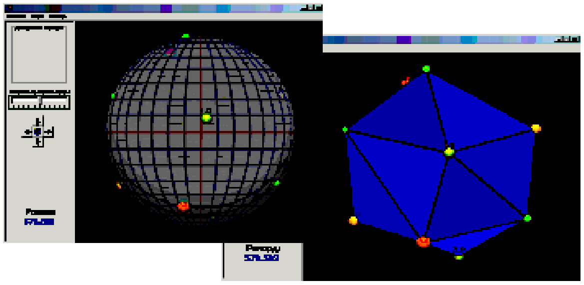
"This thesis shall explain the details of a method called complex tracing, and lay the foundations of Dynamic Geometry, a new field of research that opened up after we solved the continuity problem for interactive geometry software. I came into this project right after I decided not to write my thesis on Cinderella, the interactive geometry software. After the first few weeks of implementing the new version of Cinderella in Java I understood why it is really hard to write "just another geometry software." I had to try to implement a dynamic geometry software in order to understand why it is difficult to create a software that "behaves as expected." It needs a mathematical theory, and it was not clear to us what to do three years ago".

6. Visualization Tools

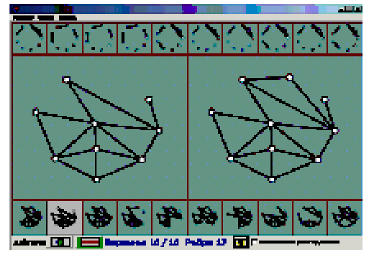
- To use problem visualization for quick understanding of essential details of a problem
- To use research results visualization to show the key solution idea

Visualization Tools for Manipulation with Unsolved Mathematical Problems (for Small N)

"Construct. Test. Explore."– contest for school in Russia based on using of computer tools.



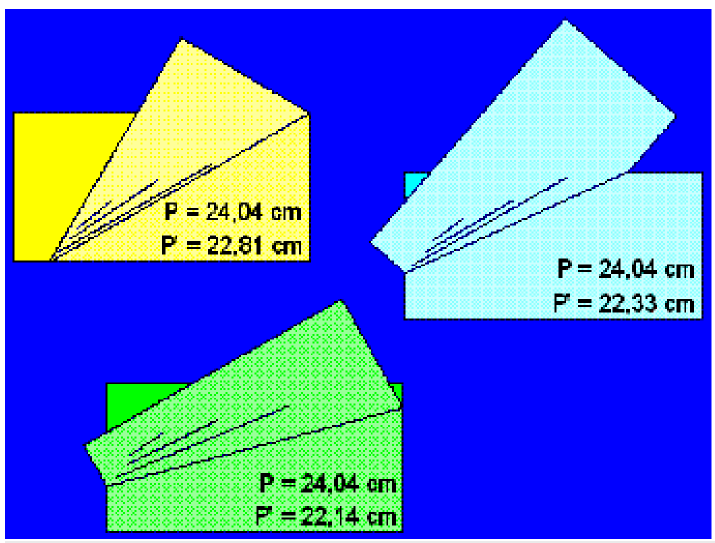
Thomson's unsolved problem: explore configurations with minimal potential energy and find global minimum.



Unsolved Graph Theory problem: reconstruct graph using set of his subgraphs of special kind.

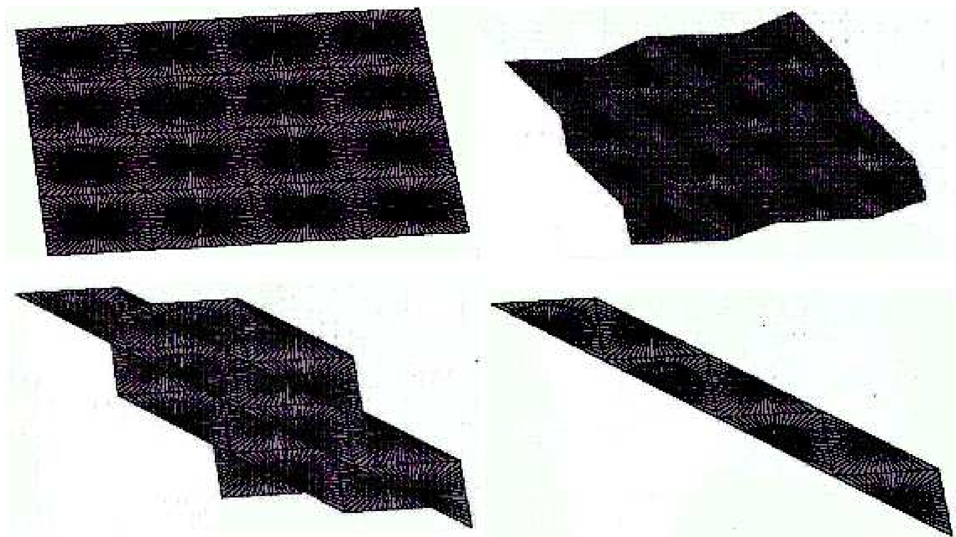
Computer Tools for Visualization of Mathematical Results

Arnold's Problem: How to bend banknote (rectangular) to increase its perimeter?

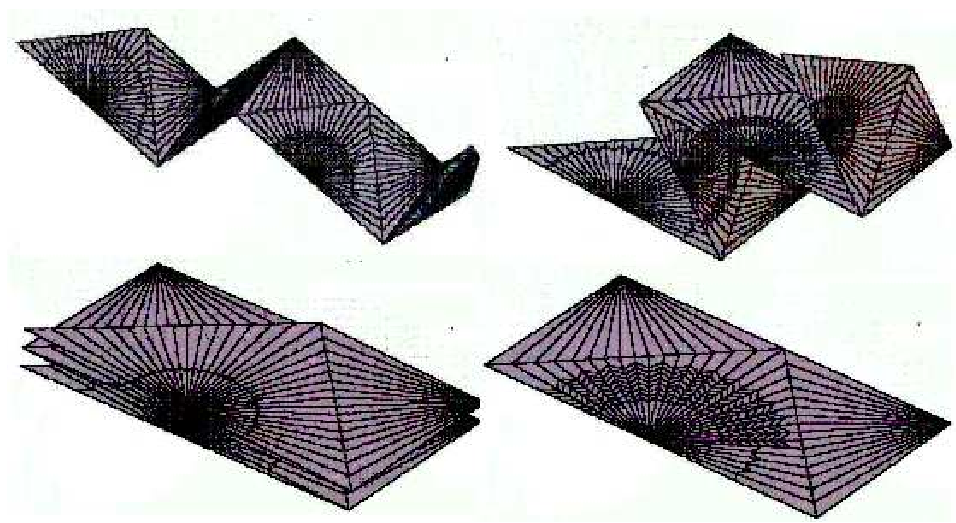


It is not so evident!
(this figure is not visualization to be discussed)

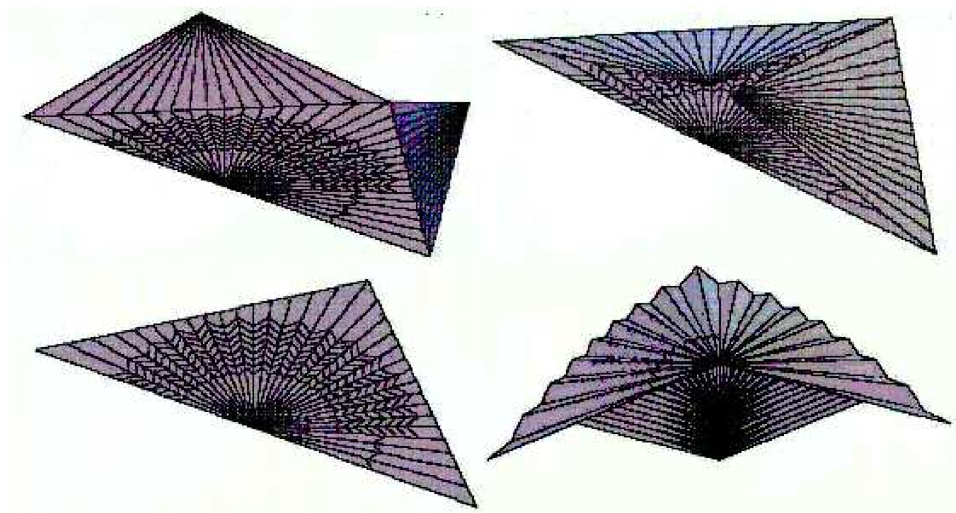
In 2004–2005, Arnold's problem was solved by young mathematician A. Tarasov from Mathematical Department of Moscow State University in his PhD thesis.



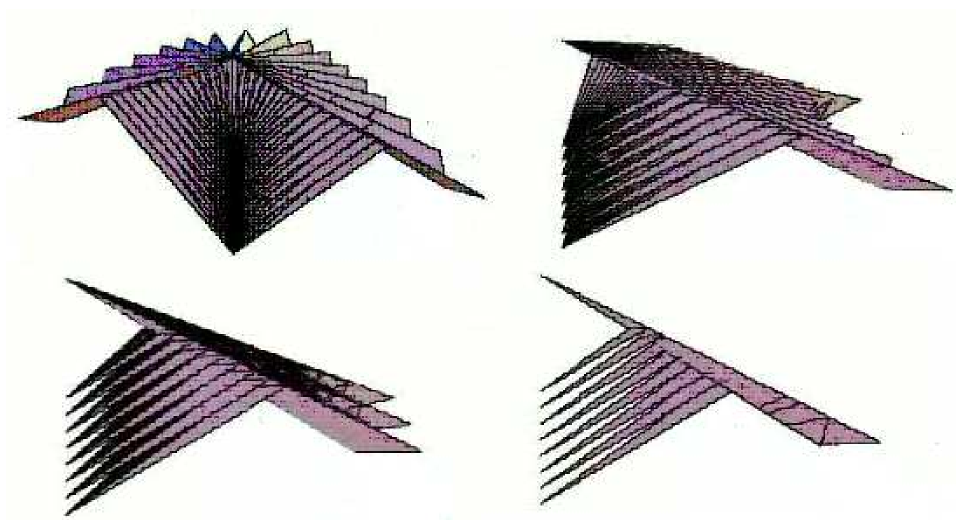
He gave a visualization of his idea in the form of computer animation.



It is not full solution but its idea



For full solution one must increase number of initial areas for bending and use a small shift between those areas to destroy symmetry



7. Tools for Publication of Research Results

Of course every mathematician needs tools for publications such as TEX and its new versions (LATEX, for example).

Another needed tool is MathML - W3C standard for Internet publication of mathematical formulas (unfortunately till now widespread browsers don't embeds it).

Thus we conclude that mathematicians must know standard tools for preparing publications.

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