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CONTROL IN DISSIPATIVE DYNAMICAL SYSTEMS: THE BASIC LAWS

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Abstract. We propose a simple mechanical model to define all the main principles of applied control theory in dissipative dynamical system.

Key words : Energy stabilization, dissipative system.

1. **Introduction**

The general idea of control in dynamics is very simple. You can change one or more parameters in dynamical equations to design the desired result, i.e. to produce some meanings of other system parameters. What kind of system you describe, what kind of laws you have in it - these features depend on physical area. From another side the problems how to organize the most effective control and what sort of methods you need to apply belong to the area of mathematical control theory. Therefore, the applied control theory in dynamical systems is an intersection of these two sciences.

The majority of physical systems are dissipative. It means that the energy in such a system is not conservative (not a constant), and because of a lot of forces (friction etc) with a time it becomes less and less. That's why the control laws in dissipative systems are so important in general applied control theory. Unfortunately, if you will start to read some introductory book, you will be involved immediately in a set of complicated examples and principles, and you will be very confused with it. To help you and to create for you an easy start is the purpose of this article.

Here we present a mechanical example of dissipative system to illustrate all three standard control schemes. This example is much more simple than the system of damping oscillator [1]. Our model does not demand any knowledge of Langrangian or Hamiltonian approach in dynamics.

We will discuss only one control goal - *stabilization*. It means that we want to achieve some *fixed* level of a physics parameter. Other goals, like tracking, synchronization etc, you can study from other books.

2. **The reachable level of the energy in dissipative systems**

In the frame of control theory for the dynamical systems we can face the set of general so-called "theorems" (and actually they are *not* theorems, because they have been derived from the investigation of some concrete systems). For the case of energy stabilization, when the control goal is to achieve some desired level of the energy value H_*, there is a sufficient difference between conservative and dissipative systems. In a dissipative system it is not possible any more to reach an arbitrary level of energy. In the Langrangian or Hamiltonian system with the mass m and the small dissipation coefficient ρ, the energy level $\bar{H}$ that we can achieve with the control limited by the constant $\bar{u}$ is estimated with the value:

$$\bar{H} = \frac{m}{2}\left(\frac{\bar{u}}{\rho}\right)^2 . \tag{1}$$

We keep the traditional notation of control theory. Here $\bar{u}$ is such a value that $u(t) \leq \bar{u}$ for any control signal u. In a conservative system in principle we can achieve *any* level of the energy.

Thus, Eq.(1) supposed to be the main law of energy stabilization in dissipative systems.

2.1. **Mechanical model.**

Let's describe a car started from the rest; it means that the initial velocity is zero: $v(0) = 0$. We involve two horizontal forces: the motor force u and the viscous friction that is proportional to the car velocity $v(t)$. Then the second Newton law can be presented in the form:

$$\dot{v} = u(v,t) - \rho \cdot v . \tag{2}$$

In Eq.(2) the positive viscousity coefficient ρ and the force u are normalized with the mass m. Of course, the friction makes our system dissipative.

2.2. **Parametric control of constant signal.**

From the point of control theory the force $u(v,t)$ is a control signal, and to specify its dependency on v and t is equivalent to choose the control scheme. Its simplest form (that in fact is not even a real control) is so-called *parametric control* with the fixed $u = \bar{u}$ =const. Then the solution of Eq.(2) with the initial condition $v(0) = 0$ is:

$$v(t) = \frac{\bar{u}}{\rho}\left(1 - e^{-\rho t}\right) . \tag{3}$$

It means that the velocity is increasing and in the limit $t \to \infty$ it reaches its maximal value $v_{max} = \bar{u}/\rho$. The reachable level of the energy is estimated with

$$\bar{H} = \frac{mv_{max}^2}{2} = \frac{m}{2}\left(\frac{\bar{u}}{\rho}\right)^2 ,$$

which coincides with (1).

2.3. **Feedforward control, or open-loop control.** Now our dissipative model (2) will be controlled with open-loop algorithm, where the control signal u is a function of the time, but not the velocity. We remind that the function u is limited by $\bar{u}$. Let's define it as a smooth function, $u(0) = 0$, and it is maximal in the limit $t \to \infty$. The model

$$u = \bar{u}\left(1 - e^{-\alpha t}\right)$$

with two positive constants $\bar{u}$ and α is quite realistic, because the motor force cannot produce the maximal level immediately. In the frame of feedforward signal we obtain the solution of (2):

$$v(t) = \frac{\bar{u}}{\rho(\rho - \alpha)}\left[\rho\left(1 - e^{-\alpha t}\right) - \alpha\left(1 - e^{-\rho t}\right)\right] . \tag{4}$$

Once again, as it was in the previous case of parametric control, the function $v(t)$ is limited for any ρ and α and $v(t) \to v_{max} = \bar{u}/\rho$ as $t \to \infty$. The law (1) is reproduced.

2.4. **Feedback control, or closed-loop control.** Let's discuss finally the most complicated and the most effective algorithm of feedback control. Now the control signal is the function of the time and the state vector of our system, i.e. the velocity, because this vector is one-dimensional in the Cauchy problem (2). To realize the closed-loop scheme we choose the speed gradient (SG) method (for more details see [2]). Let's the desired level of the energy is H_*. The purpose of the SG method is to minimize the goal function

$$Q = \frac{1}{2}(H_0 - H_*)^2 ,$$

where H_0 is the present energy level of non-controlled system. In our simple case $H_0 = mv^2/2$. SG represents the control signal u with the time derivative of the goal function $\dot{Q}$. In the case of proportional feedback with some positive coefficient γ, it is defined in the form:

$$u = -\gamma\frac{\partial \dot{Q}}{\partial u} .$$

We can calculate:

$$\dot{Q} = (H_0 - H_*) \cdot \dot{H}_0 = (H_0 - H_*) \cdot mv \cdot \dot{v}$$

with $\dot{v}$ from RHS(2). Finally:

$$u = -\gamma m v \cdot (H_0 - H_*) \ .$$

The car energy should increase:

$$\dot{H}_0 = mv \cdot \dot{v} = mv \cdot (u - \rho v) \geq 0 \ .$$

We have $v \geq 0$, then $v \leq u/\rho$. For the limited signal $|u| \leq \bar{u}$ we get $v \leq \bar{u}/\rho$, and Eq.(1) is true.

Additionally we can conclude that the constant coefficient γ is limited too. Indeed, H_0 belongs to the interval $[0, H_*]$, and the control signal is limited:

$$|u| = \gamma m (H_* - H_0) v \leq \bar{u} \ .$$

Therefore,

$$\gamma^2 m (H_* - H_0)^2 H_0 \leq \frac{\bar{u}^2}{2} \ .$$

The maximum of LHS of the last inequality corresponds to $H_0 = H_*/3$, thus,

$$\gamma \leq \frac{\bar{u}}{\sqrt{m}} \left(\frac{2}{3} H_* \right)^{-3/2} \ . \tag{5}$$

The estimation (5) completely coinsides with the result of [1], calculated in the frame of other physics model: controlled ocsillators. It is also a general law of SG algorithm control in dissipative systems.

3. Control time estimation

Now we apply our model (2) to estimate the control time t_*. For simplicity it can be expressed with the desired meaning of the velocity v_* corresponding to $H_* = mv_*^2/2$. Then the control time t_* is defined as:

$$\frac{|v(t_*) - v_*|}{v_*} \leq \varepsilon \tag{6}$$

with a fixed small dimensionless parameter $\varepsilon << 1$.

3.1. Parametric and feedforward control. For some open-loop scheme we have to put $\bar{u} = \rho v_*$. Then (3) becomes

$$v(t) = v_* (1 - e^{-\rho t}) \ ,$$

and we get

$$t_* = \frac{-\ln \varepsilon}{\rho} \ . \tag{7}$$

For more complicated open-loop scheme (4), if we suppose $\alpha >> \rho$, the same estimation (7) is true.

3.2. **Feedback control.** For the same case the goal function is:

$$Q = \frac{1}{2}(v - v_*)^2 \ ,$$

and

$$u = u_0 - \frac{\partial \dot{Q}}{\partial u}$$

with a constant u_0 (without it we cannot achieve our goal of the velocity stabilization, please, check it!). The solution in the frame of SG method is possible only with the choice:

$$u_0 = \gamma v_* \ ,$$

then

$$v(t) = v_*(1 - e^{-(\gamma+\rho)t}) \ .$$

Thus, in SG method the control time defined by (6) is different:

$$t_* = \frac{-\ln\varepsilon}{\gamma + \rho} \ . \tag{8}$$

Comparing (8) with (7) we can conclude that the control time in closed-loop schemes is *less* than in open-loop schemes. Thus, feedforward control is more effective.

4. **Further reading**

Now you should be more familiar with the main ideas of control in dissipative systems, and you can study other examples of physics systems like [1]. For more details on applied conrol and SG method see [2]. If you want to study the theory of applied control seriously, we strongly recommend you the books [3] and [4].

References

[1] A. Fradkov. Exploring nonlinearity by feedback. *Physica D* **128** (1999), pp. 159-168.
[2] S. Borisenok. Modern control theory and its application. *Math Track* **1** (2005), pp. 72–80.
[3] A. L. Fradkov, and A. Yu. Pogromsky. Introduction to Control of Oscillations and Chaos. Singapore: World Scientific, 1999.
[4] J. Zabczyk. Mathematical Control Theory: An Introduction. Springer, 1992.