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COST MINIMIZATION IN ECONOMICS: MAINTENANCE OF MACHINES BY A TEAM OF REPAIRMEN (MACHINE INTERFERENCE PROBLEM)

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There is a set of n similar working machines which break down from time to time independently of each other. They are repaired by a team of $r (r \leq n)$ repairmen, while every repairman can serve simultaneously not more then one machine and each machine is served again only by one repairman. The repaired machine operates as before damage. Its uninterrupted working time is a random variable and exponentially distributed with parameter λ, its repair time is also random, distributed by the exponential distribution with parameter μ.

Let a, b, w denote the expected number (the average) of machines working, being served and waiting for service, respectively,at a given time in a steady-state condition.

$a-$average number of operating machines	$b-$ average number of machines being served	$w-$average number of machines waiting for service

Obviously $a + b + w = n$.

Now b is also the average number of busy repairmen. Denote by $\bar{b}$ the average number of unoccupied repairmen (that is of repairmen idle)

$b-$ average number of busy repairmen	$\bar{b}-$ average number of idle (unoccupied) repairmen

Obviously $b+\bar{b}=r,$ and $\bar{b}=r-b.$
Now we introduce the cost function for the characterization of the quality of the system's operation

$$L(r,n)=c_1.w+c_2.\bar{b}$$

where c_1 is a loss per hour of a machine standing idle in the queue awaiting repair and c_2-a loss per hour (a wage per hour to be paid for hiring a repairman) due to standing idle of the repairman.

The cost function expresses the operational losses and it is natural to try to reduce these losses at minimum, that is to find the optimal number of repairmen $\tilde{r}$ to be hired at which

$$L(\tilde{r},n)=\min L(r,n), \quad 0\leq r\leq n$$

cost function = operational losses = $L(r,n)=c_1.w+c_2.\bar{b}.$
$\tilde{r}$ $-$the optimal number of repairmen reducing losses at minimum
$L(\tilde{r},n)=\min L(r,n), \quad 0\leq r\leq n$

Let us introduce now the so-called operative efficiency of the system $\eta=\frac{b}{r}$, that is the ratio of the average number of busy repairmen to the total number of repairmen. All main characteristics of the system $a,b,\bar{b},w$ can be expressed through η. In particular

$\frac{\bar{b}}{n}=\frac{r}{n}.(1-\eta), \quad \frac{w}{n}=1-\frac{r}{n.p}.\eta,$ where $p=\lambda/(\lambda+\mu), \quad q=1-p=\mu/(\lambda+\mu).$

We have found(using the theory of the Birth and Death processes and the limit theorems of probability theory)that if the ratio $\frac{r}{n}$ is close to the number $p=\lambda/(\lambda+\mu),$ then the operative efficiency η turns out to be close to 1, which means that nearly all the repairmen are busy serving the machines being out of order and at the same time the number of machines in the queue awaiting for repair is negligible as compared to the total number n of machines in the system.

This intuitively unexpected fact is far from obvious and it has led us to the conclusion that for the optimal number of repairmen $\tilde{r}$, both losses $c_1.w$ and $c_2.\bar{b}$ might be simultaneously minimized – a happy and rare event in economic world analysis. And we have bravely continued our search for the optimal number $\tilde{r}$ of repairmen which reduces the operational losses at minimum.

We have to note here that seemingly simply formulated real-world economical problems are quite difficult to solve analytically and if ever solved usually this is done by involving the powerful mathematical techniques.

For the solution of the above mentioned optimization problem we were forced to prove new type limit theorem in probability theory based on which we have introduced the following function

$$\psi(x) = \big(\frac{c_1}{p} + c_2\big) \cdot \frac{x \cdot \Phi(x) + \phi(x)}{\Phi(x) + \frac{1}{\sqrt{p}} \cdot \Phi(-\frac{x}{\sqrt{p}}) \cdot \exp\big(\frac{q \cdot x^2}{2p}\big)} - \frac{c_1}{p} \cdot x$$

(Note that: $\psi(-\infty) = \psi(+\infty) = +\infty$),

where $\phi(x)$ is the density and $\Phi(x)$−the distribution function of the famous Gaussian (standard normal) random variable. This function may be quite sympathetic for Statisticians but not for Economists.

Now let x^* be a point of the minimum of the function $\psi(x)$ over the real line $(-\infty, +\infty)$ and introduce the following number

$$\boxed{\tilde{r} = \Big[n \cdot p + x^* \cdot \sqrt{npq}\Big]}$$

where the square brackets denote the integer part. Then $\tilde{r}$ turns out to be the optimal (speaking rigorously asymptotically optimal) number of repairmen to be hired!

Thus we have solved analytically the cost-minimization problem related to the machine interference problem.

References

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