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WHAT IS MATHEMATICAL BILLIARD?

U. A. ROZIKOV

Institute of Mathematics and Information Technologies, Tashkent
Uzbekistan.
Email: rozikovu@yandex.ru

Introduction

The game of billiards is played on a rectangular table (known as a billiard table) upon which balls are placed. One ball (the "cue ball") is then struck with the end of a "cue" stick, causing it to bounce into other balls and reflect off the sides of the table. Real billiards can involve spinning the ball so that it does not travel in a straight line, but the mathematical study of billiards generally consists of reflections in which the reflection and incidence angles are the same. However, strange table shapes such as circles and ellipses are often considered. In the Fig.1 some examples of mathematical billiard table and trajectories are shown.

Mathematical theory of chaotic billiards was born in 1970 when Ya. Sinai published his seminal paper [1]; and now it is only 38 years old. But during these years it grew and developed at a remarkable speed, and became a well-established and ourishing area within the modern theory of dynamical systems and statistical mechanics.

Many interesting problems can arise in the detailed study of billiards trajectories. For example, any smooth plane convex set has at least two double normals, so there are always two distinct "to and from"paths for any smoothly curved table. Analysis of billiards path can involve sophisticated use of ergodic theory and dynamical systems.

One can also consider billiard paths on polygonal billiard tables. The only closed billiard path of a single circuit in an acute triangle is the pedal triangle. There are an infinite number of multiple-circuit paths, but all segments are parallel to the sides of the pedal triangle. There exists a closed billiard path inside a cyclic quadrilateral if its circum center lies inside the quadrilateral (Wells 1991).

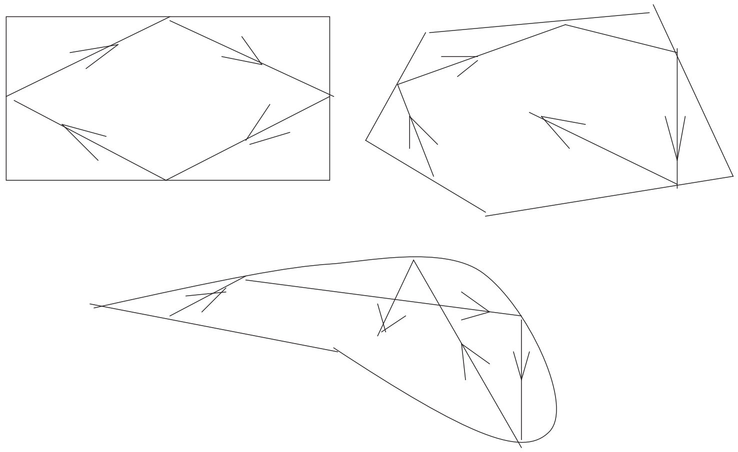

Figure 1

Notable billiard tables are:

Hadamard's billiards. Hadamard's billiards concern the motion of a free point particle on a surface of constant negative curvature, in particular, the simplest compact Riemann surface with negative curvature, a surface of genus 2 (a two-holed donut). The model is exactly solvable, and is given by the geodesic flow on the surface. It is the earliest example of deterministic chaos ever studied, having been introduced by Jacques Hadamard in 1898.

Artin's billiards. Artin's billiards concern the free motion of a point particle on a surface of constant negative curvature, in particular, the simplest non-compact Riemann surface, a surface with one cusp. The billiards are notable in being exactly solvable, and being not only ergodic but also strongly mixing. Thus they are an example of an Anosov system. Artin billiards were first studied by Emil Artin in 1924.

Sinai's billiards. The table of the Sinai billiard is a square with a disk removed from its center; the table is flat, having no curvature. The billiard arises from studying the behavior of two interacting disks bouncing inside a square, reflecting off the boundaries of the square and off each other. By eliminating the center of mass as a configuration variable, the dynamics of two interacting disks reduces to the dynamics in the Sinai billiard.

The billiard was introduced by Yakov G. Sinai as an example of an interacting Hamiltonian system that displays physical thermodynamic properties: it is ergodic and has a positive Lyapunov exponent. As a model of a classical gas, the Sinai billiard is sometimes called the Lorentz gas.

Sinai's great achievement with this model was to show that the classical Boltzmann-Gibbs ensemble for an ideal gas is essentially the maximally chaotic Hadamard billiards.

This note attempts to present the very elementary notions of the mathematical theory of billiards. We give most interesting and popular applications of the theory. It starts with the most elementary examples. Main part of this note is translated from book [3] and taken from [2].

There are several exercises. The reader is strongly encouraged to do exercises when reading the note, as this is the best way to grasp the main concepts and eventually master the techniques of (elementary) billiard theory.

Pouring problems

Problem 1. *There are two vessels with capacities 7 and 11 liter and there is a greater of a flank filled by water. How to measure by these vessels exactly 2 liter of water?*

Solution. In the problem the billiard table can be considered as a parallelogram (see Fig. 2).

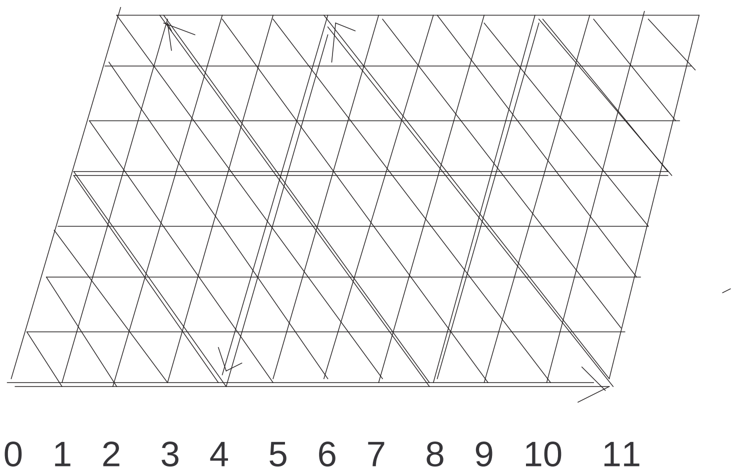

Figure 2.

The sides of the table must be 7 and 11. Following the trajectory showed in Fig.2 we can conclude following:

– The ball starts its trajectory at the point $(0,0)$ (the left bottom vertex). This position of the ball means that both vessels are empty.

– In the next position it goes to $(0,11)$ this means the big vessel is full and the small vessel is empty.

– Then it goes to he position $(7,4)$ which mens that water has been poured from the large vessel to the small one.

– The next position $(0,4)$ corresponds to the act that the small vessel has been poured out.

We should continue to follow the trajectory until one of the vessels will contain exactly 2 liters of water. Then the described algorithm gives the solution of the problem.

Remark. The algorithm described above is not the shortest one. The shortest algorithm can be obtained if you will first direct the ball to point $(7, 0)$ (the left top vertex).

Exercise 1. a) There are two vessels with capacities 7 and 11 liter and greater of a flank filled by water. How to measure by these vessels exactly 3 liter of water?

b) There are two vessels with capacities 5 and 12 liters and greater of a flank filled by water. How to measure by these vessels exactly 4 liter of water.

Problem 2 *There is a vessel with capacity 8, which is full of water. There are two empty vessels with capacities 3 and 5 liter. How to pour the water in two greater vessels equally (i.e. both vessels must contain exactly 4 liter of water)?*

Solution. The table for this problem is a 3×5 parallelogram (see Fig.3).

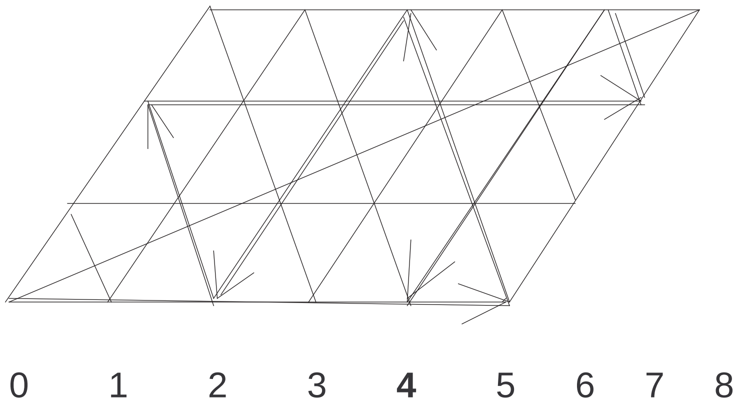

Figure 3.

The large diagonal of the parallelogram, which corresponds to the vessel with capacity 8, is divided to 8 partes by the inclined straight lines. Following the trajectory, showed in Fig.3 we should go until it separates 4 liter.

Remark. If two smaller vessels have coprime (relatively prime) capacities (i.e. the capacity (volume) numbers have not a common divisor $\neq 1$) and the biggest vessel has a capacity lager (or equal) than sum of the capacities of the smaller vessels then using these three vessels one can measure water with liters: from 1 until the capacity of the mid vessel. For example, if there are three vessels with capacities 12,13 and 26 respectively. Then one can measure l liter of water for any $l \in \{1, 2, ..., 13\}$.

Exercise 2. a) Solve the analogue of problem 2 for vessels with capacities 3, 4 and 7 respectively.

b) Given three vessels with capacities 7, 9 and 12. Take a table like in Fig.4 and check that the ball can go to any point 1,2,...,9 except 6.

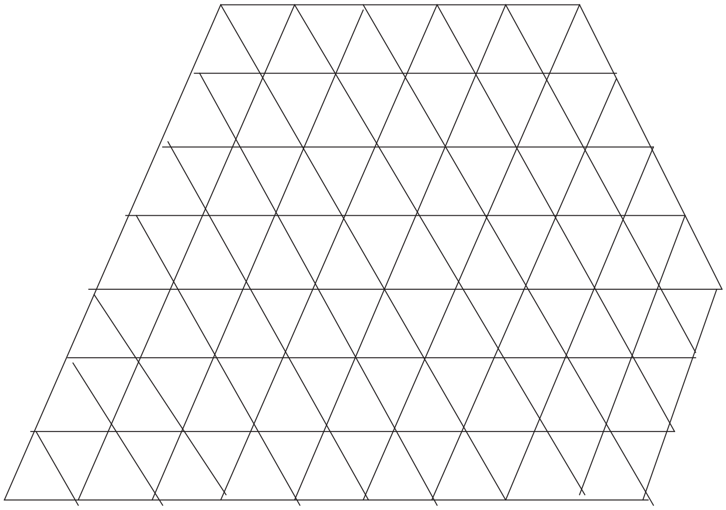

Figure 4

Billiard in the Circle

Although a unit circle is a very simple figure, there are a few interesting things one can say about the billiard inside it (see [2] for details). The circle enjoys rotational symmetry, and a billiard trajectory is completely determined by the angle α made with the circle. This angle remains the same after each reflection. Each consecutive impact point is obtained from the previous one by a circle rotation through angle $\theta = 2\alpha$.

If $\theta = \frac{2\pi p}{q}$, then every billiard orbit is q-periodic and makes p turns about the circle; one says that the *rotation number* of such an orbit is $\frac{p}{q}$. If θ is not a rational multiple of π, then every orbit is infinite. The first result on π-irrational rotations of the circle is due to Jacobi. Denote the circle rotation through angle θ by T_θ.

Theorem 1. *If θ is π-irrational, then the T_θ-orbit of every point is dense. In other words, every interval contains points of this orbit.*

Exercise 3. Prove that if θ is π-irrational, then the T_θ-orbit has infinitely many points in any arc Δ of the circle.

Let us continue the study of the sequence $x_n = x+n\theta \ \ mod\, 2\pi$ with π-irrational θ. If $\theta = \frac{2\pi p}{q}$, this sequence consists of q elements which are distributed in the circle very regularly. Should one expect a similar regular distribution for π-irrational θ?

The adequate notion is that of *equidistribution* (or *uniform* distribution). Given an arc I, let $k(n)$ be the number of terms in the sequence $x_0, ..., x_{n-1}$ that lie in I. The sequence is called equidistributed on the circle if

$$\lim_{n\to\infty} \frac{k(n)}{n} = \frac{|I|}{2\pi}, \quad \text{for every } I.$$

The next theorem is due to Kronecker and Weyl; it implies Theorem 1.

Theorem 2. *If θ is π-irrational, then the sequence $x_n = x + n\theta \ \ mod\, 2\pi$ is equidistributed on the circle.*

Now we shall give some applications of Theorems 1 and 2.

Problem 3. *Distribution of first digits. Consider the sequence*

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...$$

consisting of consecutive powers of 2.

Can a power of 2 start with 2008?

Is a term in this sequence more likely to start with 3 or 4?

Solution. Let us consider the second question: 2^n has the first digit k if, for some non-negative integer q, one has

$$k10^q \leq 2^n < (k+1)10^q.$$

Take logarithm base 10:

$$\log k + q \leq n \log 2 < \log(k+1) + q. \tag{1}$$

Since q is of no concern to us, let us consider fractional parts of the numbers involved. Denote by $\{x\}$ the fractional part of the real number x. Inequalities (1) mean that $\{n \log 2\}$ belongs to the interval $I = [\log k, \log(k+1))$. Note that $\log 2$ is an irrational number (why?). Thus by Theorem 1 there is a number n_0 such that $2^{n_0} = k....$ Using Theorem 2, we obtain the following result.

Corollary. *The probability $p(k)$ for a power of 2 to start with digit k equals* $\log(k+1) - \log k$.

The values of these probabilities are approximately as follows: $p(1) = 0.301$, $p(2) = 0.176, p(3) = 0.125, p(4) = 0.097, p(5) = 0.079, p(6) = 0.067$, $p(7) = 0.058, p(8) = 0.051, p(9) = 0.046$.

We see that $p(k)$ monotonically decreases with k; in particular, 1 is about 6 times as likely to be the first digit as 9.

Exercise 4. a) What is the distribution of the first digits in the sequence $2^n C$ where C is a constant?

b) Find the probability that the first m digits of a power of 2 is a given combination $k_1 k_2 ... k_m$.

c) Investigate similar questions for powers of other numbers.

d) Prove that if p is such that $p \neq 10^q$ (for some $q = 1, 2, ...$) then the sequence $p, p^2, p^3, ...$ has a term with the first m digits is a given combination $k_1 k_2 ... k_m$.

Remark. Surprisingly, many real life sequences enjoy a similar distribution of first digits! This was first noted in 1881 in a 2-page article by American astronomer S. Newcomb. This article opens as follows: That the ten digits do not occur with equal frequency must be evident to any one making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones. The first significant figure is oftener 1 than any other digit, and the frequency diminishes up to 9.

Problem 4. *Is there a natural number n such that* $\sin n < 10^{-2008}$?

Solution. The answer is "exists"! To prove this consider a billiard on a circle with radius 1, which corresponds to the rotation number $\theta = 1$ radian (see Fig.5). Then sequence $\sin 0, \sin 1, \sin 2, ...$ on $[-1, 1]$ corresponds to the trajectory $0, 1, 2, ...$ of the billiard with the starting point 0. Since 1 radian is π-irrational, by Theorem 1 we get the result. Note that the question is trivial if one considers $x \in R$ instate of $n = 1, 2, ...$

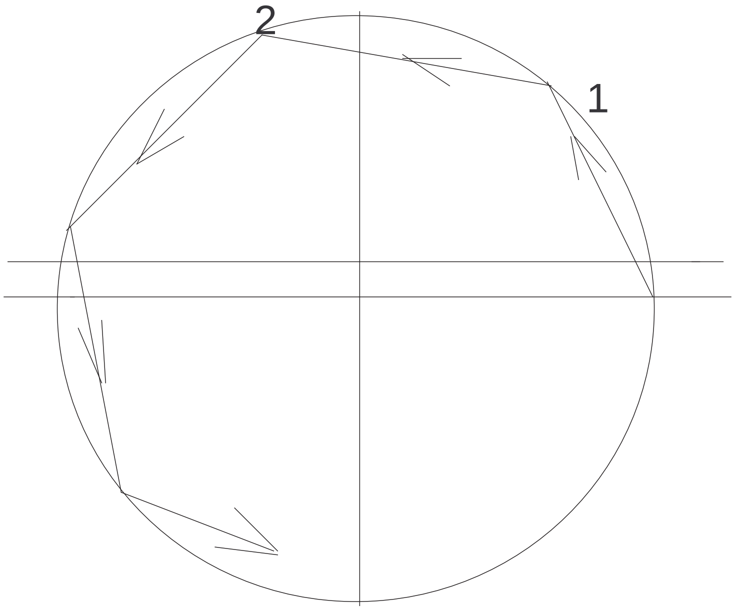

Figure 5

Exercise 5. a) Is there a natural number n such that $\cos n < 10^{-2008}$?

b) Is there a natural number n such that $\tanh n > 10^{2008}$?

References

[1] Ya. G. Sinai, Dynamical systems with elastic reactions. Ergodic properties of dispersing billiards, *Russ. Math. Surv.*, **25**, 137-189, (1970).

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