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SEMIGROUPS OF LINEAR OPERATORS ON LOCALLY CONVEX SPACES

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Abstract. Semigroups of linear operators acting on locally convex spaces have been studied by many authors. In this paper we consider semigroups of special linear operators on locally convex spaces. The paper is organized in three sections. In the first section some properties of series on m-convex algebras are recalled. In the second section we point out two remarkable subalgebras of $\mathcal{L}$(X). The third section is dedicated to semigroups of linear operators. A few properties of semigroups are given in this context. The main result of this section is a Hille - Yosida type theorem for "C_0-bounded semigroups".

Key words : Semigroup, bounded, locally bounded, infinitesimal generator

1. A few remarks on M- convex algebras

In the following we denote by N^*, the set of all natural numbers toghether with 0, i.e. $N^* = N \cup \{0\}$.

Definition 1.1. *Let Y be a real or complex Hausdorff locally convex algebra whose topology is given by a directed family of seminorms*
$Q = (q_\alpha),\ \alpha \in A$. We say that (Y, Q) is m-convex algebra if for every $\alpha \in A$ and $x,\ y \in Y$, the following inequality holds:
$q_\alpha(xy) \le q_\alpha(x) q_\alpha(y)$.

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Proposition 1.2. Let (Y,Q) be a sequentially complete m-convex algebra and $u_n,\ v_n \in Y,\ n \in N^*$, two sequences. Consider also
$w_n = \sum\limits_{k=0}^{n} u_k v_{n-k},\ n \in N^*$.

Assume that the series $\sum\limits_{n\geq 0} u_n$ and $\sum\limits_{n\geq 0} v_n$ are absolutely convergent (i.e $\sum\limits_{n\geq 0} q_\alpha(u_n)$ and $\sum\limits_{n\geq 0} q_\alpha(v_n)$ are convergent for all $\alpha \in A$).

Then the following statements are true:

1. $\sum\limits_{n\geq 0} w_n$ is absolutely convergent.
2. $\sum\limits_{n\geq 0} w_n = \left(\sum\limits_{n\geq 0} u_n\right)\left(\sum\limits_{n\geq 0} v_n\right)$.

Proof. At first we see that $\sum\limits_{n\geq 0} u_n$ and $\sum\limits_{n\geq 0} v_n$ are convergent. Let $\alpha \in A$. We denote $s_n = \sum\limits_{i=0}^{n} q_\alpha(u_i)$ and $t_n = \sum\limits_{j=0}^{n} q_\alpha(v_j)$. Obviously we have for each $n \in \mathbf{N}^*$.

$\left(\sum\limits_{i=0}^{n} u_i\right)\left(\sum\limits_{j=0}^{n} v_j\right) - \sum\limits_{i=0}^{n} w_i = u_1 v_n + ... + u_i v_j + ... + u_n v_n$ where $i + j > n$.

Moreover we can write:

$$q_\alpha\left[\left(\sum_{i=0}^{n} u_i\right)\left(\sum_{j=0}^{n} v_j\right) - \sum_{i=0}^{n} w_i\right] \leq q_\alpha(u_1)q_\alpha(v_n) + ... + q_\alpha(u_i)q_\alpha(v_j) + ...$$

$$+ q_\alpha(u_n)q_\alpha(v_n) \leq \left(\sum_{\left[\frac{n}{2}\right]<i\leq n} q_\alpha(u_i))\right)\left(\sum_{j=0}^{n} q_\alpha(v_j)\right) +$$

$$\left(\sum_{\left[\frac{n}{2}\right]<j\leq n} q_\alpha(v_j)\right)\left(\sum_{i=0}^{n} q_\alpha(u_i)\right) = \left(s_n - s_{\left[\frac{n}{2}\right]}\right) t_n + \left(t_n - t_{\left[\frac{n}{2}\right]}\right) s_n.$$

Hence it follows that $\sum\limits_{n\geq 0} w_n = \left(\sum\limits_{n\geq 0} u_n\right)\left(\sum\limits_{n\geq 0} v_n\right)$.

In order to prove that $\sum\limits_{n\geq 0} w_n$ is absolutely convergent we apply the first part of the proof in the following context:
$Y = \mathbf{R}$, $q_\alpha(u_n),\ q_\alpha(v_n),\ n \in \mathbf{N}$. If $a_n = q_\alpha(u_0)q_\alpha(v_n) + ... + q_\alpha(u_n)q_\alpha(v_0)$ then $\sum\limits_{n\geq 0} a_n$ is convergent. On the other hand, $q_\alpha(w_n) \leq a_n$, which proves that $\sum\limits_{n\geq 0} w_n$ is absolutely convergent.□

Theorem 1.3. *Let $(Y,\ Q)$, be a sequentially complete m - convex algebra with unit* 1. *Then the following statements are true:*

1. $\sum\limits_{n\geq 0}\dfrac{x^n}{n!}$ *is convergent for all* $x\in Y$ *and we denote* $e^x=\sum\limits_{n\geq 0}\dfrac{x^n}{n!}$.
2. *If* $x,\ y\in Y$ *and* $xy=yx$, *then* $e^{x+y}=e^xe^y$.
3. $\lim\limits_{t\to 0}\dfrac{e^{tx}-1}{t}=x$, *for all* $x\in Y$.
4. $\dfrac{d}{dt}(e^{tx})=xe^{tx}=e^{tx}x$, *for all* $t\in\mathbf{R}$ *and* $x\in Y$.

Proof.

1. Let $\alpha\in A$ and $x\in Y$. Then $q_\alpha\left(\dfrac{x^n}{n!}\right)\leq\dfrac{q_\alpha^n(x)}{n!}$, which ensures the convergence of $\sum\limits_{n\geq 0}\dfrac{x^n}{n!}$.

2. Let $x,y\in Y$ such that $xy=yx$. We know from the first point that $e^{x+y}=\sum\limits_{n\geq 0}\dfrac{(x+y)^n}{n!}$ and $\dfrac{(x+y)^n}{n!}=\dfrac{1}{n!}\sum\limits_{p=0}^{n}C_n^{n-p}x^py^{n-p}$
$=\dfrac{1}{n!}\sum\limits_{p=0}^{n}\dfrac{n!}{(n-p)!p!}x^py^{n-p}=\sum\limits_{p=0}^{n}\dfrac{x^p}{p!}\dfrac{y^{n-p}}{(n-p)!}$.

Now it is enough to apply Proposition 1.2 to $u_n=\dfrac{x^n}{n!}$ and $v_n=\dfrac{y^n}{n!}$. Then $w_n=\sum\limits_{p=0}^{n}\dfrac{x^p}{p!}\dfrac{y^{n-p}}{(n-p)!}$, and consequently $e^{x+y}=e^xe^y$.

3. Let $t\in\mathbf{R},\ t\neq 0$ and $x\in Y$. We also have

$$\frac{e^{tx}-1}{t}-x=t\left(\frac{x^2}{2!}+\frac{tx^3}{3!}+...+\frac{t^{n-2}x^n}{n!}+...\right).$$

Denote by $\sigma_n(t)=\dfrac{x^2}{2!}+\dfrac{tx^3}{3!}+...+\dfrac{t^{n-2}x^n}{n!}$. Then $\lim\limits_{t\to 0}t\sigma_n(t)=\dfrac{e^{tx}-1}{t}-x$.
On the other hand, for each $\alpha\in A$ it follows that:
$q_\alpha\left(t\sigma_n(t)\right)=|t|\,q_\alpha\left(\sigma_n(t)\right)\leq|t|\left(\dfrac{q_\alpha^2(x)}{2!}+|t|\,\dfrac{q_\alpha^3(x)}{3!}+...+|t|^{n-2}\,\dfrac{q_\alpha^n(x)}{n!}\right)$.
Suppose that $|t|\leq 1$.

Then $q_\alpha\left(t\sigma_n(t)\right)\leq|t|\,e^{q_\alpha(x)}$ and $q_\alpha\left(\dfrac{e^{tx}-1}{t}-x\right)\leq|t|\,e^{q_\alpha(x)}$, which implies that $\lim\limits_{t\to 0}\dfrac{e^{tx}-1}{t}=x$.

4. Let $t\in\mathbf{R}$ is fixed and $h\in\mathbf{R}$.

$$\frac{1}{h}\left(e^{(t+h)x}-e^{tx}\right)=\frac{1}{h}\left(e^{tx}e^{hx}-e^{tx}\right)=\frac{1}{h}e^{tx}\left(e^{hx}-1\right)$$
$$=e^{tx}\frac{e^{hx}-1}{h}=\frac{e^{hx}-1}{h}e^{tx}.$$

Hence it follows that $\frac{d}{dt}\left(e^{tx}\right) = e^{tx}x = xe^{tx}$ and the proof is complete. □

Remark 1.4. If $x \in Y$ and $\alpha \in A$ such that $q_\alpha(I) = 1$, then $q_\alpha(e^x) \leq e^{q_\alpha(x)}$.

2. Special Algebras of Operators

Throughout this paper X is a real or complex Hausdorff locally convex space, whose topology is given by the directed family of seminorms $P = (p_\alpha),\ \alpha \in A$. We denote by $\mathcal{L}(X)$ the space of all linear and continuous operators on X.

Definition 2.1. *We set:*

$$\mathcal{L}_0(X) := \left\{ \begin{array}{c} T \in \mathcal{L}(X) \mid \forall \alpha \in A,\ \exists M(\alpha) > 0,\ \forall x \in X, \\ p_\alpha\left(T(x)\right) \leq M(\alpha)p_\alpha(x) \end{array} \right\} \tag{1}$$

$$\mathcal{L}_F(X) := \left\{ \begin{array}{c} T \in \mathcal{L}(X) \mid \exists M > 0,\ \forall (x \in X,\ \alpha \in A)\ , \\ p_\alpha\left(T(x)\right) \leq Mp_\alpha(x) \end{array} \right\} \tag{2}$$

For each $\alpha \in A$ we define $q_\alpha : \mathcal{L}_0(X) \to \mathbf{R}$ as follows:

$$q_\alpha(T) = \sup\left\{\frac{p_\alpha\left(T(x)\right)}{p_\alpha(x)} \mid x \in X,\ p_\alpha(x) \neq 0\right\} \tag{3}$$

We define also $\|,\| : L_F(X) \to \mathbf{R}$ as follows

$$\|T\| = \sup_{\alpha \in A} q_\alpha(T) \tag{4}$$

Proposition 2.2. The following assertions are true:

1. $(q_\alpha),\ \alpha \in A$ is a sufficient family of seminorms on $\mathcal{L}_0(X)$.
2. $\mathcal{L}_0(X)$ is m - convex algebra.
3. If in addition X is a sequentially complete, then $\mathcal{L}_0(X)$ is also sequentially complete.

Proof.

1. Let $T \in \mathcal{L}_0(X)$ and $\alpha \in A$. Then, according to (3), we can write $p_\alpha\left(T(x)\right) \leq q_\alpha(T)p_\alpha(x)$, for all $x \in X$. Therefore if $q_\alpha(T) = 0$, for all $\alpha \in A$, then $T = 0$.

2. Let $T, U \in \mathcal{L}_0(X)$ and $\alpha \in A$. Then from the first point we conclude that $TU \in \mathcal{L}_0(X)$ and $q_\alpha(TU) \leq q_\alpha(T)q_\alpha(U)$.

3. Let us suppose that X is sequentially complete. Let $T_n \in \mathcal{L}_0(X),\ n \in \mathbf{N}$, be a Cauchy sequence. Then $T_n(x),\ n \in \mathbf{N}$ is a Cauchy sequence in X for all $x \in X$. Denote by $T(x) = \lim_{n\to\infty} T_n(x)$. Let $\varepsilon > 0$ and $\alpha \in A$. Then there exists $N(\varepsilon, \alpha) \in \mathbf{N}$ such that $q_\alpha(T_{n+p} - T_n) < \varepsilon$ for all $n \geq N(\varepsilon, \alpha)$ and $p \in \mathbf{N}$, and consequently $p_\alpha\left(T_{n+p}(x) - T_n(x)\right) \leq \varepsilon p_\alpha(x)$ for all $x \in X$. Now, passing to the limit as $p \to \infty$ in the above inequality we get $p_\alpha\left(T(x) - T_n(x)\right) \leq \varepsilon p_\alpha(x)$

for all $x \in X$ and $n \geq N(\varepsilon, \alpha)$. Hence it follows that $T \in \mathcal{L}_0(X)$ and $T = \lim\limits_{n\to\infty} T_n$ in $\mathcal{L}_0(X)$, because $q_\alpha(T - T_n) < \varepsilon$ for all $n \geq N(\varepsilon, \alpha)$. □

Proposition 2.3. The following assertions are true:

1. $\mathcal{L}_F(X)$ is a normed algebra.
2. If in addition X is sequentially complete, then $\mathcal{L}_F(X)$ is a Banach algebra.

Proof.

1. It is clear that $\mathcal{L}_F(X)$ is vector space and the positive map defined by (4) is a norm. Moreover, if $T, U \in \mathcal{L}_F(X)$ then $TU \in \mathcal{L}_F(X)$ and $\|TU\| \leq \|T\| \, \|U\|$.

2. Let $T_n \in \mathcal{L}_F(X),\ n \in \mathbf{N}$, be a Cauchy sequence. Then $T_n,\ n \in \mathbf{N}$ is also a Cauchy sequence in $\mathcal{L}_0(X)$ (see (4)). By using Proposition 2.2 it follows that $T_n,\ n \in \mathbf{N}$ is convergent in $\mathcal{L}_0(X)$ and denote by T its limit. Let $\varepsilon > 0$. There exists $N(\varepsilon) \in \mathbf{N}$ such that $\|T_{n+p} - T_n\| < \varepsilon$ for all $n \geq N(\varepsilon)$ and $p \in \mathbf{N}$. Hence it follows that:

$$q_\alpha\left(T_{n+p} - T_n\right) < \varepsilon, \text{ for all } \alpha \in A,\ n \geq N(\varepsilon) \text{ and } p \in \mathbf{N}. \tag{5}$$

By passing to the limit as $p \to \infty$ in (5) we get $q_\alpha\left(T - T_n\right) < \varepsilon$ for all $\alpha \in A,\ n \geq N(\varepsilon)$. This means that $\|T - T_n\| < \varepsilon$ for all $n \geq N(\varepsilon)$. Therefore, $T \in \mathcal{L}_F(X)$ and $\lim\limits_{n\to\infty} T_n = T$ in $\mathcal{L}_F(X)$. □

Example 2.4. Let $X = C(\mathbf{R})$ be the space of the real continuous functions equipped with the following family of seminorms:
$p_n(f) = \sup\left\{|f(x)|,\ |x| \leq n\right\},\ n \in \mathbf{N}$ and $f \in X$. Let also $T : X \to X$ be defined by the following formula:

$$T(f)(x) = \begin{cases} \int\limits_0^x f(t)dt, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

It is clear that T is a linear operator and $p_n\left(T(f)\right) \leq np_n(f)$ for all $f \in X$ and $n \in \mathbf{N}$. In conclusion $T \in \mathcal{L}_0(X)$ and $q_n(T) \leq n$ for all $n \in \mathbf{N}$.

Example 2.5. Let $X = C(\mathbf{R})$ equipped with the same topology as above and $U : X \to X$ defined by the following formula

$$U(f)(x) = \begin{cases} e^{-x}\int\limits_0^x f(t)dt, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Let $n \in \mathbf{N}$ and $x \in (0, n]$. Then $|U(f)(x)| \leq e^{-x}\int\limits_0^x |f(t)|\, dt$
$\leq xe^{-x}p_n(f) \leq p_n(f)$. Therefore $p_nU(f) \leq p_n(f)$ for all $n \in \mathbf{N}$ and for all $f \in X$. This means that $U \in \mathcal{L}_F(X)$ and $\|U\| \leq 1$.

3. $\mathbf{C_0}$ - Semigroups

Definition 3.1. *A mapping $T : [0, \infty) \to \mathcal{L}(X)$ is called a semigroup if the following conditions are satisfied:*

1. $T(t+s) = T(t)T(s)$, *for all* $t, s \in [0, \infty)$.
2. $T(0) = 1$.

If in addition $\lim\limits_{t \to t_0} T(t)(x) = T(t_0)(x)$ *for all* $t_0 \in [0, \infty)$ *and* $x \in X$, *we say that T is a C_0-semigroup.*

We denote by $D = \left\{ x \in X \mid \lim\limits_{t \to 0} \dfrac{T(t)(x) - x}{t} \text{ exists} \right\}$ *and consider the linear operator $U : D \to X$ defined by* $U(x) = \lim\limits_{t \to 0} \dfrac{T(t)(x) - x}{t}$. *The operator U is called the infinitesimal generator of T and D is usually denoted by $D(U)$.*

Proposition 3.2. Let $T : [0, \infty) \to \mathcal{L}_0(X)$ be a C_0-semigroup. Then for all $x \in X$, $\alpha \in A$ and $l > 0$, there exists $M(x, \alpha, l) > 0$, such that $p_\alpha (T(t)(x)) \leq M(\alpha, x, l)$, for all $t \in [0, l]$.

Proof. Let $x \in X$, $\alpha \in A, l > 0$ and $\varepsilon > 0$. Then there exists $\delta = \delta(x, \alpha, l) > 0$, such that $p_\alpha (T(h)(x) - x) < \varepsilon$, for all $h \in [0, \delta]$. Consider $N_0 \in \mathbf{N}$, such that $N_0 \delta > l$. Let also $0 \leq t \leq l$ and $k = \left[\dfrac{t}{\delta} \right]$. This means that $k\delta \leq t < (k+1)\delta$ and $k < N_0$. Let $h < \delta$, such that $t = k\delta + h$. Then $T(t)(x) = T(k\delta)T(h)(x) = T^k(\delta)T(h)(x)$ and $p_\alpha (T(t)(x)) \leq q_\alpha^k (T(\delta)) p_\alpha (T(h)(x)) \leq q_\alpha^k (T(\delta)) (\varepsilon + p_\alpha(x)) \leq \max\limits_{k < N_0} q_\alpha^k (T(\delta)) (\varepsilon + p_\alpha(x)) = M(x, \alpha, l)$. □

Proposition 3.3. Let $T : [0, \infty) \to \mathcal{L}_0(X)$ be a C_0-semigroup. Then for each $\alpha \in A$ and $x \in X$, there exist $M(\alpha, x) > 0$ and $\omega(\alpha) \geq 0$, such that $p_\alpha (T(t)(x)) \leq M(\alpha, x)e^{\omega(\alpha)t}$, for all $t \geq 0$.

Proof. Let $\alpha \in A$ and $x \in X$. Let also $t > 0$ and $k \in \mathbf{N}$, such that $k \leq t < k+1$. Then there exists $h \in [0, 1)$, such that $t = k + h$ and $T(t)(x) = T(k)T(h)(x) = T^k(1)T(h)(x)$. From Proposition 3.2, we know that there exists $M(\alpha, x) \geq 0$, such that $p_\alpha (T(u)(x)) \leq M(\alpha, x)$, for all $u \in [0, 1]$. Therefore it follows that

$$p_\alpha (T(t)(x)) \leq q_\alpha^k (T(1)) p_\alpha (T(h)(x)) \leq M(\alpha, x) q_\alpha^k (T(1)).$$

If $q_\alpha (T(1)) \geq 1$, we conclude that $p_\alpha (T(t)(x)) \leq M(\alpha, x) q_\alpha^t (T(1))$. If we denote by $\omega(\alpha) = \ln q_\alpha (T(1))$, it follows that $p_\alpha (T(t)(x)) \leq M(\alpha, x)e^{\omega(\alpha)t}$, for all $t \geq 0$, and the proof is complete. □

Definition 3.4. *A semigroup $T : [0, \infty) \to \mathcal{L}_0(X)$ is called locally bounded, if for each $\alpha \in A$ and $l > 0$, there exists $M(\alpha, l) \geq 0$, such that $q_\alpha (T(t)) \leq M(\alpha, l)$, for all $t \in [0, l]$. A semigroup $T : [0, \infty) \to \mathcal{L}_0(X)$ is called bounded, if for each $\alpha \in A$, there exists $M(\alpha) > 0$, such that $q_\alpha (T(t)) \leq M(\alpha)$, for all $t \geq 0$.*

Proposition 3.5. Let $T : [0,\infty) \to \mathcal{L}_0(X)$ be a semigroup. Then the following statements are equivalent:

1. T is locally bounded.

2. For each $\alpha \in A$, there exists $M(\alpha) \geq 1$ and $\omega(\alpha) > 0$, such that $q_\alpha(T(t)) \leq M(\alpha)e^{\omega(\alpha)t}$, for all $t \geq 0$.

Proof. Let us suppose that T is locally bounded. Let $\alpha \in A$ and $M(\alpha) > 0$, such that $q_\alpha(T(t)) \leq M(\alpha)$, *for all* $t \in [0,1]$. Let now
$t > 1,\ k \in \mathbf{N}$, such that $k \leq t < k+1$ and $h < 1$, with the property
$t = k+h$. Then $T(t) = T(k)T(h) = T^k(1)T(h)$ and $q_\alpha(T(t)) \leq M(\alpha)q_\alpha^k(T(1))$
$\leq M(\alpha)q_\alpha^t(T(1))$, (if $q_\alpha(T(1)) \geq 1$). If we denote by $\omega(\alpha) = \ln q_\alpha(T(1))$, it follows that $q_\alpha(T(t)) \leq M(\alpha)e^{\omega(\alpha)t}$, for all $t \geq 0$. The implication 2⇒1 is obvious. □

Theorem 3.6. *Let (X,P) be a sequentially complete locally convex space. Suppose that $T : [0,\infty) \to \mathcal{L}(X)$ is a C_0-semigroup and that $U : D(U) \to X$ is its generator. Then the following assertion are true:*

1. *For $t \geq 0$ and $x \in X$, it follows that* $\lim\limits_{h\to 0}\frac{1}{h}\int\limits_t^{t+h} T(s)(x)ds = T(t)(x)$.

2. *For $t \geq 0$ and $x \in X$,* $\int\limits_0^t T(s)(x)ds \in D(U)$ *and* $U\left(\int\limits_0^t T(s)(x)ds\right) = T(t)(x) - x$.

3. *For $x \in D(U)$ and $t \geq 0$, $T(t)(x) \in D(U)$ and $UT(t)(x) = T(t)U(x)$.*

4. *If in addition, $T(t) \in \mathcal{L}_0(X)$, for all $t \geq 0$ and T is locally bounded, then* $\frac{d}{dt}T(t)(x) = T(t)U(x) = UT(t)(x)$ *and* $\int\limits_s^t T(u)U(x)du = T(t)(x) - T(s)(x)$, *for all $x \in D(U),\ s,t \geq 0$.*

Proof.

1. Let $x \in X,\ t > 0,\ \alpha \in A$ and $\varepsilon > 0$. Then from the continuity of the map $s \to T(s)(x)$, there exists $\delta(\alpha,\varepsilon) > 0$, such that if $|s-t| < \delta(\alpha,\varepsilon)$, then $p_\alpha(T(s)(x) - T(t)(x)) < \varepsilon$. On the other hand,,

$$p_\alpha\left(\frac{1}{h}\int\limits_t^{t+h} T(s)(x)ds - T(t)(x)\right) = p_\alpha\left(\frac{1}{h}\int\limits_t^{t+h}(T(s)(x) - T(t)(x))\,ds\right) \leq$$

$\frac{1}{|h|}\left|\int\limits_t^{t+h} p_\alpha(T(s)(x) - T(t)(x))\,ds\right| < \varepsilon$, for all $h \in \mathbf{R}$, such that $|h| < \delta(\alpha,\varepsilon)$

2. Let $x \in X,\ t \geq 0$ and $h > 0$. Then we have:

$$\frac{1}{h}\left(T(h)\int\limits_0^t T(s)(x)ds - \int\limits_0^t T(s)(x)ds\right) = \frac{1}{h}\int\limits_0^t T(s+h)(x)ds - \frac{1}{h}\int\limits_0^t T(s)(x)ds.$$

For $h + s = \xi$, the above equality becomes:

$\frac{1}{h}\int_0^t T(s+h)(x)ds - \frac{1}{h}\int_0^t T(s)(x)ds = \frac{1}{h}\int_h^{t+h} T(\xi)(x)d\xi - \frac{1}{h}\int_0^t T(\xi)(x)d\xi$
$= \frac{1}{h}\int_h^{h+t} T(\xi)(x)d\xi - \frac{1}{h}\int_0^{t+h} T(\xi)(x)d\xi + \frac{1}{h}\int_t^{t+h} T(\xi)(x)d\xi = \frac{1}{h}\int_t^{t+h} T(\xi)(x)d\xi -$
$\frac{1}{h}\int_0^h T(\xi)(x)d\xi$

Using the first point we conclude that:

$$\lim_{h\to 0}\frac{1}{h}\left(T(h)\left(\int_0^t T(s)(x)ds\right) - \int_0^t T(s)(x)ds\right) = T(t)(x) - x.$$

3. Let $x \in D(U)$, $t > 0$ and $h > 0$. Then one gets:

$$\lim_{h\to 0}\frac{1}{h}\left(T(h)T(t)(x) - T(t)(x)\right) = \lim_{h\to 0} T(t)\frac{T(h)(x)-x}{h} = T(t)U(x) = UT(t)(x).$$

4. It is clear that the forward derivative $\frac{d}{dt}T(t)(x)$ equals $T(t)U(x)$, for each $x \in D(U)$.

Now we show that the backward derivative equals also $T(t)U(x)$. Let $x \in D(U)$, $t > 0$, $h > 0$ and $\alpha \in A$. Since T is locally bounded, there exists $M(\alpha, t) \geq 1$, such that $q_\alpha(T(u)) \leq M(\alpha, t)$, for all $u \in [0, t]$. Then we have:

$p_\alpha\left(\frac{1}{h}\left(T(t)(x) - T(t-h)(x)\right) - T(t)U(x)\right) =$
$p_\alpha\left(\frac{1}{h}\left(T(t-h)T(h)(x) - T(t-h)(x)\right) - T(t-h)T(h)U(x)\right) =$
$p_\alpha\left(T(t-h)\left(\frac{T(h)(x)-x}{h} - T(h)U(x)\right)\right) \leq$
$q_\alpha\left(T(t-h)\right)p_\alpha\left(\frac{T(h)(x)-x}{h} - T(h)U(x)\right) \leq$
$M(\alpha, t)p_\alpha\left(\frac{T(h)(x)-x}{h} - T(h)U(x)\right)$

Since $\lim_{h\to 0}\left(\frac{T(h)(x)-x}{h} - T(h)U(x)\right) = 0$, it follows that $\frac{d}{dt}T(t)(x) = T(t)U(x)$ and consequently $\frac{d}{dt}T(t)(x) = T(t)U(x) = UT(t)(x)$.

Let now $s, t \geq 0$ and $x \in D(U)$. Then $\int_s^t T(u)U(x)du = T(t)(x) - T(s)(x)$, because the map $u \to T(u)(x)$ is a primitive of the map $u \to T(u)U(x)$. $\square$

Corollary 3.7. Let (X, P) be a sequentially complete locally convex space. Suppose that $T : [0, \infty) \to \mathcal{L}_0(X)$ is a locally bounded C_0-semigroup and $U : D(U) \to X$ be its generator. Then the following assertions hold:

1. $\overline{D(U)} = X$.
2. U is closed.

Proof. 1. Let $x \in X$ and $0 < h_n \to 0$. From Theorem 3.6, it follows that $\int_0^{h_n} T(s)(x)ds \in D(U)$ and $\lim_{n\to\infty} \frac{1}{h_n} \int_0^{h_n} T(s)(x)ds = x$, which means that $x \in \overline{D(U)}$.

2. Let now $t > 0$, $x, y \in X$, and a generalized sequence $x_\delta \in D(U)$, $\delta \in \Delta$, such that $\lim_{\delta\in\Delta} (x_\delta, U(x_\delta)) = (x, y)$. Using again Theorem 3.6 we can write

$$\int_0^t T(s)U(x_\delta)ds = T(t)(x_\delta) - x_\delta, \text{ for all } \delta \in \Delta. \tag{6}$$

Let $\alpha \in A$ and $M(\alpha, t) > 0$, such that $q_\alpha (T(s)) \leq M(\alpha, t)$, for all $s \in [0, t]$. We have also $p_\alpha \left(\int_0^t (T(s)U(x_\delta) - T(s)(y)) \, ds \right) \leq \int_0^t p_\alpha (T(s)(U(x_\delta) - y)) \, ds$. On the other hand,,

$$p_\alpha (T(s)(U(x_\delta) - y)) \leq q_\alpha (T(s)) \, p_\alpha (U(x_\delta) - y) \leq M(\alpha, t) p_\alpha (U(x_\delta) - y) \, .$$

Thus $p_\alpha \left(\int_0^t T(s)U(x_\delta)ds - \int_0^t T(s)(y)ds \right) \leq tM(\alpha, t)p_\alpha (U(x_\delta) - y)$.

In conclusion $\lim_{\delta\in\Delta} \int_0^t T(s)U(x_\delta)ds = \int_0^t T(s)(y)ds$.

Passing to the limit in (6) as $\delta \in \Delta$, we obtain the following equality: $\int_0^t T(s)(y)ds = T(t)(x) - x$, which is equivalent to

$$\frac{1}{t} \int_0^t T(s)(y)ds = \frac{T(t)(x) - x}{t}. \tag{7}$$

By Theorem 3.6 and (7) we conclude that $x \in D(U)$ and $U(x) = y$. Therefore U is a closed linear operator. □

Proposition 3.8. Let (X, P) be a sequentially complete locally convex space and let $T, S : [0, \infty) \to \mathcal{L}_0(X)$ be two locally bounded C_0-semigroups. If T and S have the same generator, then $T = S$.

Proof. Let $U : D(U) \to X$ be the joint generator of T and S, $t > 0$ and $x \in D(U)$. Consider $f : [0, t] \to X$, defined by $f(s) = T(t - s)S(s)(x)$.By Theorem 3.6, it follows that $f'(s) = -T(t-s)US(s)(x)+T(t-s)US(s)(x) = 0$, which implies that $S(t)(x) = T(t)(x)$. Since U is densely defined it follows that $T = S$. $\square$

Lemma 3.9. *Let (X, P) be a sequentially complete locally convex space. Suppose that $f : [0, \infty) \to X$ is a continuous map which verifies the following property:*
For each $\alpha \in A$, there exists $g(\alpha) : [0, \infty) \to \mathbf{R}$, such that $p_\alpha(f(x)) \leq g(\alpha)(x)$, for all $x \geq 0$. If $\int_0^\infty g(\alpha)(x)dx$ is convergent, it follows that $\int_0^\infty f(x)dx$ is also convergent.

Proof. Let $\alpha \in A$ and $\varepsilon > 0$. Then there exists $\delta(\alpha, \varepsilon) > 0$, such that $\left|\int_{t'}^{t''} g(\alpha)(x)dx\right| \leq \varepsilon$, for all $t', t'' > \delta$. On the other hand, we have

$$p_\alpha\left(\int_{t'}^{t''} f(x)dx\right) \leq \left|\int_{t'}^{t''} p_\alpha(f(x))\, dx\right| \leq \left|\int_{t'}^{t''} g(\alpha)(x)dx\right| < \varepsilon.$$

Therefore $\int_0^\infty f(x)dx$ is convergent. $\square$

Proposition 3.10. Let (X, P) be a sequentially complete locally convex space, $T : [0, \infty) \to \mathcal{L}_0(X)$ is a bounded C_0-semigroup and $U : D(U) \to X$ is its generator. Then the following assertions hold:

1. $\int_0^\infty e^{-\lambda t} T(t)(x)dt \in D(U)$, for all $x \in D(U)$ and $\lambda > 0$.
2. $U\left(\int_0^\infty e^{-\lambda t} T(t)(x)dt\right) = \int_0^\infty e^{-\lambda t} T(t)U(x)dt$.

Proof. Let $\alpha \in A$, $x \in X$ and $\lambda > 0$. There exists $M(\alpha) > 0$, such that $q_\alpha(T(t)) \leq M(\alpha)$, for all $t \geq 0$. Hence it follows that:

$$p_\alpha\left(e^{-\lambda t} T(t)(x)\right) \leq e^{-\lambda t} q_\alpha(T(t))\, p_\alpha(x) \leq M(\alpha) e^{-\lambda t} p_\alpha(x),$$

for all $t \geq 0$. According to Lemma3.9, one deduces the convergence of $\int_0^\infty e^{-\lambda t} T(t)(x)dt$.

Let $x \in D(U)$, $t > 0$, $\lambda > 0$ and $f : [0, t] \to X$, defined by
$f(s) = e^{-\lambda s} T(s)(x)$.

From Theorem 3.6, it follows that $f(s) \in D(U)$, $U(f(s)) = e^{-\lambda s} T(s)U(x)$ and U is closed. Hence it follows that
$U\left(\int_0^t e^{-\lambda s} T(s)(x)ds\right) = \int_0^t e^{-\lambda s} T(s)U(x)ds$, and by passing to the limit as $t \to \infty$, we conclude finally that
$\surd U\left(\int_0^\infty e^{-\lambda s} T(s)(x)ds\right) = \int_0^\infty e^{-\lambda s} T(s)U(x)ds$. $\square$

Definition 3.11. *A mapping $R : (0,\infty) \to \mathcal{L}(X)$ is called a resolvent, if the following condition is satisfied*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu), \text{ for all } \lambda, \mu \in (0,\infty). \tag{8}$$

Lemma 3.12. *Let $R : (0,\infty) \to \mathcal{L}_0(X)$ be a resolvent. Suppose that for each $\alpha \in A$, there exists $M(\alpha) > 0$, such that $q_\alpha(\lambda R(\lambda)) \leq M(\alpha)$, for all $\lambda > 0$. Then R is indefinite deriviable and the following formula holds:*

$$\frac{d^n}{d\lambda^n} R(\lambda) = (-1)^n n! \left(R(\lambda)\right)^{n+1}, \text{ for all } \lambda > 0 \text{ and } n \geq 1. \tag{9}$$

Proof. At first we are proving that for each natural number $n \geq 1$, the mapping $\lambda \to R^n(\lambda)$ is continuous. From (8) we deduce $q_\alpha(R(\mu) - R(\lambda)) \leq |\lambda - \mu| \frac{M(\alpha)}{\lambda} \frac{M(\alpha)}{\mu}$, for all $\alpha \in A$, which proves that $\lim\limits_{\mu\to\lambda} R(\mu) = R(\lambda)$. Let now $n \geq 2$ and $\lambda, \mu \in (0,\infty)$. Then the following equality holds:

$$R^n(\mu) - R^n(\lambda) = (R(\mu) - R(\lambda))\left(R^{n-1}(\mu) + R^{n-2}(\mu)R(\lambda) + ... + R^{n-1}(\lambda)\right).$$

For each $\alpha \in A$ we get

$$q_\alpha\left(R^n(\mu) - R^n(\lambda)\right) \leq \left(\frac{(M(\alpha))^{n-1}}{\mu^{n-1}} + ... + \frac{(M(\alpha))^{n-1}}{\lambda^{n-1}}\right) q_\alpha\left(R(\mu) - R(\lambda)\right),$$

which proves that $\lim\limits_{\mu\to\lambda} R^n(\mu) = R^n(\lambda)$. Let now $\lambda, \mu > 0$, $\mu \neq \lambda$. As a consequence of (8) we have $\frac{R(\mu) - R(\lambda)}{\mu - \lambda} + (R(\lambda))^2 = R(\lambda)\left(R(\lambda) - R(\mu)\right)$. Thus $\frac{d}{d\lambda} R(\lambda) = -(R(\lambda))^2$. Using the above decomposition and (9) the following equalities are valid:

$$\frac{1}{\mu - \lambda}\left(\frac{d^n}{d\lambda^n}R(\mu) - \frac{d^n}{d\lambda^n}R(\lambda)\right) = \frac{(-1)^n n!}{\mu - \lambda}\left(R^{n+1}(\mu) - R^{n+1}(\lambda)\right) =$$

$$(-1)^n n! \frac{R(\mu) - R(\lambda)}{\mu - \lambda}\left((R(\mu))^n + (R(\mu))^{n-1} R(\lambda) + ... + (R(\lambda))^n\right). \tag{10}$$

By passing to the limit as $\mu \to \lambda$ in (10) we conclude that:

$$\frac{d^{n+1}}{d\lambda^{n+1}} R(\lambda) = (-1)^n n! \left(-R^2(\lambda)\right)(n+1)R^n(\lambda) = (-1)^{n+1}(n+1)! R^{n+2}(\lambda).$$

which means that (9) is valid. □

Definition 3.13. *Let $U : D(U) \to X$ be a linear operator. We denote by*

$$\rho(U) = \left\{\lambda \in \mathbf{C} \mid \lambda I - U \text{ is one-to-one and } (\lambda I - U)^{-1} \in \mathcal{L}(X)\right\}.$$

The mapping $R(\cdot, U) : \rho(U) \to \mathcal{L}(X)$ defined by $R(\lambda, U) = (\lambda I - U)^{-1}$ is a resolvent and is called the resolvent of U. The set of complex numbers $\rho(U)$

is called the resolvent set of U*. We also denote by*
$\rho_0(U) = \{\lambda \in \rho(U) \mid R(\lambda, U) \in \mathcal{L}_0(X)\}$.

Lemma 3.14. *Let* $U : D(U) \to X$ *be a linear operator such that:*

1. $\overline{D(U)} = X$.
2. $(0, \infty) \subset \rho_0(U)$.
3. *For each* $\alpha \in A$*, there exists* $M(\alpha) > 0$*, such that* $q_\alpha(\lambda R(\lambda, U)) \leq M(\alpha)$*, for all* $\lambda > 0$*. Then the following assertions hold:*

1. U *is closed.*
2. $\lim_{\lambda \to \infty} \lambda R(\lambda, U)(x) = x$*, for all* $x \in X$.
3. $D(U) = \left\{ x \in X \mid \lim_{\lambda \to \infty} \lambda\left(\lambda R(\lambda, U)(x) - x\right) \text{ exists} \right\}$ *and* $\lim_{\lambda \to \infty} \lambda\ (\lambda R(\lambda, U)(x) - x) = U(x)$*, for all* $x \in D(U)$.

Proof.

1. The closeness of U is a simple consequence of the second hypothesis.

2. Let at first $x \in D(U)$ and $\lambda > 0$. From Definition 3.13, one can deduce the following equalities:

$$R(\lambda, U)(\lambda I - U)(x) = x, \text{ for all } x \in D(U), \tag{11}$$

$$(\lambda I - U)R(\lambda, U)(x) = x, \text{ for all } x \in X, \tag{12}$$

$$R(\lambda, U)U(x) = UR(\lambda, U)(x), \text{ for all } x \in D(U). \text{ Let } \alpha \in A \text{ and } x \in D(A). \tag{13}$$

From (11) it follows that

$$p_\alpha\left(\lambda R(\lambda, U)(x) - x\right) = p_\alpha\left(R(\lambda, U)U(x)\right)$$
$$\leq q_\alpha\left(R(\lambda, U)\right) p_\alpha\left(U(x)\right) \leq \frac{M(\alpha)}{\lambda} p_\alpha\left(U(x)\right).$$

and consequently $\lim_{\lambda \to \infty} \lambda R(\lambda, U)(x) = x$.

Let now $x \in X$, $\alpha \in A$ and $x_\delta \in D(U)$, $\delta \in \Delta$, such that $\lim_{\delta \in \Delta} x_\delta = x$. The following equality can be written as:

$$\lambda R(\lambda, U)(x) - x = \lambda R(\lambda, U)(x) - \lambda R(\lambda, U)(x_\delta) + \lambda R(\lambda, U)(x_\delta) - x_\delta + x_\delta - x.$$

In addition we have:

$$p_\alpha\left(\lambda R(\lambda, U)(x) - x\right) \leq q_\alpha\left(\lambda R(\lambda, U)\right) p_\alpha\left(x - x_\delta\right) + p_\alpha\left(\lambda R(\lambda, U)(x_\delta) - x_\delta\right) +$$
$$+ p_\alpha\left(x_\delta - x\right) \leq M(\alpha) p_\alpha\left(x - x_\delta\right) + p_\alpha\left(\lambda R(\lambda, U)(x_\delta) - x_\delta\right) + p_\alpha\left(x_\delta - x\right).$$

Hence it follows that $\lim_{\lambda \to \infty} \lambda R(\lambda, U)(x) = x$, and the second point is proved.

3. Let $x \in D(U)$. By (11) and the second point we deduce that $\lim_{\lambda \to \infty} \lambda\left(\lambda R(\lambda, U)(x) - x\right) = U(x)$. Let now $x \in X$, such that $\lim_{\lambda \to \infty} \lambda\left(\lambda R(\lambda, U)(x) - x\right) = y$. From (12) it follows that $\lambda\left(\lambda R(\lambda, U)(x) - x\right) = U\left(\lambda R(\lambda, U)(x)\right)$. On one hand $\lim_{\lambda \to \infty} \lambda R(\lambda, U)(x) = x$, and on the other hand,

$\lim\limits_{\lambda\to\infty} U\left(\lambda R(\lambda,U)(x)\right) = y$. Because U is closed, we conclude that $x \in D(U)$ and $y = U(x)$. $\square$

Remark 3.15. Usually the mapping $\lambda \to \lambda\left(\lambda R(\lambda,U) - I\right)$ is called the Yosida approximation and it is denoted by $U(\lambda) = \lambda\left(\lambda R(\lambda,U) - I\right)$, $\lambda \in \rho(U)$. Therefore according to Proposition 3.14 we have $\lim\limits_{\lambda\to\infty} U(\lambda)(x) = U(x)$, for all $x \in D(U)$.

Lemma 3.16. *Let (X,P) be a sequentially complete locally convex space and let $U : D(U) \to X$ be a linear operator satisfying:*

1. $\overline{D(U)} = X$.
2. $(0,\infty) \subset \rho_0(U)$.
3. *For each $\alpha \in A$, there exists $M(\alpha) \geq 1$, such that $q_\alpha\left(\lambda R(\lambda,U)\right)^n \leq M(\alpha)$, for all $\lambda > 0$ and $n \geq 1$.*

Then the following assertions hold:

1. $q_\alpha(e^{tU(\lambda)}) \leq M(\alpha)$, *for all* $\alpha \in A,\ t \geq 0$ *and* $\lambda > 0$
2. $p_\alpha(e^{tU(\mu)}(x) - e^{tU(\lambda)}(x)) \leq t\left(M(\alpha)\right)^2 p_\alpha(U(\mu)(x) - U(\lambda)(x))$, *for all* $\alpha \in A,\ t \geq 0,\ \lambda,\mu \in (0,\infty)$ *and* $x \in X$.

Proof.

1. According to Proposition 2.2, $\mathcal{L}_0(X)$ is a sequentially complete m-convex algebra and from Theorem 1.3, we conclude that $e^{U(\lambda)} \in \mathcal{L}_0(X)$, for all $\lambda > 0$. Let $t \geq 0$ and $\alpha \in A$. According to Remark 3.15, we have $e^{tU(\lambda)} = e^{t\lambda^2 R(\lambda,U) - \lambda t I}$ and hence it follows that

$$q_\alpha(e^{tU(\lambda)}) \leq e^{-\lambda t} q_\alpha(e^{t\lambda^2 R(\lambda,U)}) \tag{14}$$

On the other hand, from the following equality:

$$e^{t\lambda^2 R(\lambda,U)} = I + \frac{t\lambda\, \lambda R(\lambda,U)}{1!} + ... + \frac{(t\lambda)^n \left(\lambda R(\lambda,U)\right)^n}{n!} + ...$$

we obtain the following inequalities:

$$q_\alpha\left(I + \frac{t\lambda\, \lambda R(\lambda,U)}{1!} + ... + \frac{(t\lambda)^n \left(\lambda R(\lambda,U)\right)^n}{n!}\right) \leq$$
$$\leq 1 + M(\alpha)\left(\frac{t\lambda}{1!} + ... + \frac{(t\lambda)^n}{n!}\right) \leq M(\alpha) e^{t\lambda},$$

for all $n \geq 1$.

Therefore we have $q_\alpha(e^{t\lambda^2 R(\lambda,U)}) \leq M(\alpha) e^{t\lambda}$ and from (14), it follows that $q_\alpha(e^{tU(\lambda)}) \leq M(\alpha)$, for all $\alpha \in A,\ t \geq 0$ and $\lambda > 0$.

2. Let $x \in X,\ t > 0$ and $\lambda,\mu \in (0,\infty)$. To prove the second assertion we consider the following mapping $g : [0,t] \to X$, defined by $g(s) = e^{(t-s)U(\lambda)} e^{sU(\mu)}(x)$. Using again Theorem 1.3 and Proposition 2.2, we conclude that g is derivable and

$$g'(s) = e^{(t-s)U(\lambda)} e^{sU(\mu)}\left(U(\mu)(x) - U(\lambda)(x)\right), \text{ for all } s \in [0,t]. \tag{15}$$

Therefore also we have:

$$\int_0^t g'(s)ds = g(t) - g(0) = e^{tU(\mu)}(x) - e^{tU(\lambda)}(x). \tag{16}$$

Let now $\alpha \in A$. From (15) and (16), it follows that

$$p_\alpha(e^{tU(\mu)}(x) - e^{tU(\lambda)}(x)) \le t\,(M(\alpha))^2\,p_\alpha(U(\mu)(x) - U(\lambda)(x))$$

and the proof is complete. □

Theorem 3.17. *Let (X, P) be a sequentially complete locally convex space, let $T : [0, \infty) \to \mathcal{L}_0(X)$ be a bounded C_0-semigroup and let $U : D(U) \to X$ be its generator. Then the following statements are true:*

1. $\overline{D(U)} = X$.
2. $(0, \infty) \subset \rho_0(U)$.
3. $R(\lambda, U)(x) = \int_0^\infty e^{-\lambda t} T(t)(x)dt$, *for all $\lambda > 0$ and $x \in X$.*
4. *For each $\alpha \in A$, there exists $M(\alpha) > 0$, such that $q_\alpha\left(\lambda R(\lambda, U)\right)^n \le M(\alpha)$, for all $n \in \mathbf{N}^*$ and $\lambda > 0$.*

Proof. The density of $D(U)$ is a consequence of Corollary 3.7. Because of the boundness of T, for each $\alpha \in A$, there exists a constant $M(\alpha) > 0$, such that $q_\alpha\left(T(t)\right) \le M(\alpha)$, for all $t \ge 0$. Proposition 3.10 allows us to define $R : (0, \infty) \to \mathcal{L}(X)$ by the following formula
$R(\lambda)(x) = \int_0^\infty e^{-\lambda t} T(t)(x)dt$ for all $x \in X$. It is clear that $R(\lambda)$ is a linear operator, for all $\lambda > 0$ and for each $\alpha \in A$ we have:

$$p_\alpha\left(R(\lambda)(x)\right) \le \int_0^\infty e^{-\lambda t} p_\alpha\left(T(t)(x)\right)dt \le \int_0^\infty e^{-\lambda t} q_\alpha(T(t)) p_\alpha(x)dt$$

$$\le M(\alpha)\int_0^\infty e^{-\lambda t} p_\alpha(x)dt = \frac{M(\alpha)}{\lambda} p_\alpha(x).$$

This proves that $R(\lambda) \in \mathcal{L}_0(X)$ and $q_\alpha\left(\lambda R(\lambda)\right) \le M(\alpha)$, for all $\lambda > 0$. If $x \in D(U)$ from Proposition 3.10, it follows that $\int_0^\infty e^{-\lambda t} T(t)(x)dt \in D(U)$ and $U\left(\int_0^\infty e^{-\lambda t} T(t)(x)dt\right) = \int_0^\infty e^{-\lambda t} T(t)U(x)dt$, which means that $UR(\lambda)(x) = R(\lambda)U(x)$. We know also from Theorem 3.6, that

$$T(t)(x) \in D(U) \text{ and } UT(t)(x) = T(t)U(x) = \frac{d}{dt}T(t)(x).$$

On the other hand, we estimate:

$UR(\lambda)(x) = \int_0^\infty e^{-\lambda t} T(t)U(x)dt = \lim_{b\to\infty} \int_0^b e^{-\lambda t} T(t)U(x)dt =$

$\lim_{b\to\infty} \int_0^b e^{-\lambda t} \frac{d}{dt} T(t)(x)dt = \lim_{b\to\infty} \left(e^{-\lambda t} T(b)x - x\right)$

$+ \lim_{b\to\infty} \lambda \int_0^b e^{-\lambda t} T(t)(x)dt = \lambda R(\lambda)(x) - x$, or equivalent:

$$(\lambda I - U)R(\lambda)(x) = x. \tag{17}$$

Because U is closed and densely defined, we also have

$$(\lambda I - U)R(\lambda)(x) = x, \text{ for all } x \in X.$$

Let now $x \in D(A)$. Then we can write
$R(\lambda)(\lambda I - U)(x) = \lambda R(\lambda)(x) - R(\lambda)U(x) = \lambda R(\lambda)(x) - UR(\lambda)(x) = \lambda R(\lambda)(x) + x - \lambda R(\lambda)(x) = x$ and hence we have

$$R(\lambda)(\lambda I - U)(x) = x, \text{ for all } x \in D(U). \tag{18}$$

From (18), it follows that $\lambda I - U$ is invertible and $R(\lambda) = (\lambda I - U)^{-1}$, for all $\lambda > 0$. In conclusion $(0, \infty) \subset \rho_0(U)$ and

$$R(\lambda, U)(x) = R(\lambda)(x) = \int_0^\infty e^{-\lambda t} T(t)(x)dt, \text{ for all } x \in X. \tag{19}$$

If we apply Lemma 3.12 to $R(\lambda, U)$, we conclude that $\frac{d^n}{d\lambda^n} R(\lambda, U) = (-1)^n n! R^{n+1}(\lambda, U)$ and

$$\frac{d^n}{d\lambda^n} R(\lambda, U)(x) = (-1)^n n! R^{n+1}(\lambda, U)(x). \tag{20}$$

for all $x \in X, \ n \in \mathbf{N}^*$ and $\lambda > 0$. On the other hand from (19), it follows:

$$\frac{d^n}{d\lambda^n} R(\lambda, U)(x) = \int_0^\infty e^{-\lambda t} (-t)^n T(t)(x)dt. \tag{21}$$

and from (20) and (21), we deduce the equality:

$$\lambda R(\lambda, U)^{n+1}(x) = \frac{\lambda^{n+1}}{n!} \int_0^\infty e^{-\lambda t} t^n T(t)(x)dt.$$

Let now $\alpha \in A$. From the last equality we get:

$$p_\alpha\left((\lambda R(\lambda, U))^{n+1}(x) \right) \le \frac{\lambda^{n+1}}{n!} M(\alpha) p_\alpha(x) \int_0^\infty e^{-\lambda t} t^n dt =$$

$$= \frac{\lambda^{n+1}}{n!} M(\alpha) p_\alpha(x) \frac{n!}{\lambda^{n+1}} = M(\alpha) p_\alpha(x).$$

and finally, it follows that $q_\alpha (\lambda R(\lambda, U))^n \leq M(\alpha)$, for all $\alpha \in A,\ n \in \mathbf{N}^*$ and $\lambda > 0$. □

Theorem 3.18. (Hille - Yosida) *Let (X, P) be a sequentially complete locally convex space and $U : D(U) \to X$ is a linear operator. Then the following assertions are equivalent:*

1. *U is the generator of a bounded C_0-semigroup $T : [0, \infty) \to \mathcal{L}_0(X)$.*
2. $\left\{ \begin{array}{l} \text{a) } \overline{D(U)} = X. \\ \text{b) } (0, \infty) \subset \rho_0(U). \\ \text{c) For each } \alpha \in A, \text{ there exists } M(\alpha) > 0 \text{ such that} \\ q_\alpha (\lambda R(\lambda, U))^n \leq M(\alpha), \text{ for all } \lambda > 0 \text{ and } n \geq 1. \end{array} \right.$

Proof. 1⇒2 is a consequence of Theorem 3.17. Conversely: Let $U : D(U) \to X$ be a linear operator verifying the conditions a), b) and c), and $x \in D(U)$. By Lemma 3.14, it follows that

$$\lim_{\lambda \to \infty} U(\lambda)(x) = U(x). \tag{22}$$

On the other hand, from Lemma 3.16, the following estimation follows

$$p_\alpha \left(e^{tU(\mu)}(x) - e^{tU(\lambda)}(x) \right) \leq t \left(M(\alpha) \right)^2 p_\alpha \left(U(\mu)(x) - U(\lambda)(x) \right). \tag{23}$$

for all $\alpha \in A$, $t \geq 0$, $\lambda, \mu \in (0, \infty)$. Since X is sequentially complete, from (22) and (23), we conclude that there exists the limit as $\lambda \to \infty$ of the mapping $\lambda \to e^{tU(\lambda)}(x)$ and is uniform on bounded intervals. Let us denote it by $T(t)(x) = \lim\limits_{\lambda \to \infty} e^{tU(\lambda)}(x)$. Since U is densely defined and $q_\alpha \left(e^{tU(\lambda)} \right) \leq M(\alpha)$, for all $\alpha \in A,\ \lambda > 0$ and $t \geq 0$, it follows that $\lim\limits_{\lambda \to \infty} e^{tU(\lambda)}(x) = T(t)(x)$,uniformly on bounded intervals, for all $x \in X$. Let now $a > 0,\ x \in X$ and $\lambda_n \to \infty$. Then $T(t)(x) = \lim\limits_{n \to \infty} e^{tU(\lambda_n)}(x)$ uniformly on $[0, a]$, which means that the mapping $t \to T(t)(x)$ is continuous on $[0, a]$ and consequently on $[0, \infty)$. On the other hand,, we have $p_\alpha \left(e^{tU(\lambda_n)}(x) \right) \leq q_\alpha \left(e^{tU(\lambda_n)} \right) p_\alpha(x) \leq M(\alpha) p_\alpha(x)$ and passing to the limit as $n \to \infty$ it follows that $p_\alpha (T(t)(x)) \leq M(\alpha) p_\alpha(x)$, for all $\alpha \in A,\ t \geq 0$ and $x \in X$. Since $T(t)$ is linear the above inequality shows that $T(t) \in \mathcal{L}_0(X)$ and $q_\alpha (T(t)) \leq M(\alpha)$, for all $\alpha \in A$ and $t \geq 0$.

Let $s, t \in [0, \infty),\ x \in X$ and $\alpha \in A$. For every $n \geq 1$, we can write
$T(t+s) - T(t)T(s)(x) = T(t+s)(x) - e^{(s+t)U(n)}(x) + e^{(s+t)U(n)}(x)$
$- e^{sU(n)} e^{tU(n)}(x) +$
$e^{tU(n)} e^{sU(n)}(x) - e^{tU(n)} T(s)(x) + e^{tU(n)} T(s)(x) - T(t)T(s)(x)$ and
$p_\alpha (T(t+s)(x) - T(t)T(s)(x)) \leq p_\alpha \left(T(t+s)(x) - e^{(s+t)U(n)}(x) \right) +$
$q_\alpha \left(e^{tU(n)} \right) p_\alpha \left(e^{sU(n)}(x) - T(s)(x) \right) + p_\alpha \left(e^{tU(n)} T(s)(x) - T(t)T(s)(x) \right)$. Since $q_\alpha \left(e^{tU(n)} \right) \leq M(\alpha)$, for all $n \geq 1$ and $t \geq 0$ and $\lim\limits_{n \to \infty} e^{tU(n)} T(s)(x) =$

$T(t)T(s)(x)$, it follows that $T(t+s)(x) = T(t)T(s)(x)$ for all $x \in X$. By the definition it follows also that $T(0) = I$. Therefore $T : [0,\infty) \to \mathcal{L}_0(X)$ is a bounded C_0-semigroup. Let us suppose that its generator is: $V : D(V) \to X$. Before proving that $V = U$ we show that:

$$\lim_{\lambda\to\infty} \int_0^a e^{tU(\lambda)} U(\lambda)(x)dt = \int_0^a T(t)U(x)dt. \tag{24}$$

for all $x \in D(U)$, where a is a positive fixed number. For each $\alpha \in A$ we estimate:

$$p_\alpha \left(\int_0^a \left(e^{tU(\lambda)} U(\lambda)(x) - T(t)U(x) \right) dt \right) \leq M(\alpha) \int_0^a p_\alpha \left(U(\lambda)(x) - U(x) \right) dt +$$

$$\int_0^a p_\alpha \left(e^{tU(\lambda)} U(x) - T(t)U(x) \right) dt.$$

Since $\lim_{\lambda\to\infty} U(\lambda)(x) = U(x)$ and $\lim_{\lambda\to\infty} e^{tU(\lambda)} U(x) = T(t)U(x)$ uniformly on $[0,a]$, the formula (24) is established. On the other hand, for $x \in D(U)$, $\lambda > 0$ and $t > 0$, we have the following estimate:

$$\int_0^t e^{sU(\lambda)} U(\lambda)(x)ds = e^{sU(\lambda)}(x) \mid_0^t = e^{tU(\lambda)}(x) - x. \tag{25}$$

From (24) and (25), it follows that $\int_0^t T(s)U(x)ds = T(t)(x) - x$, and moreover:

$$\frac{1}{t} \int_0^t T(s)U(x)ds = \frac{T(t)(x) - x}{t}. \tag{26}$$

From Theorem 3.6 and (26) it follows that $x \in D(V)$, $D(U) \subset D(V)$ and $V(x) = U(x)$. By Theorem 3.17, we deduce that V has the same properties as U, thus $(0,\infty) \in \rho_0(V)$ and consequently $(I - V)^{-1} \in \mathcal{L}_0(X)$.

Let $x \in X$. Then there exists $y \in D(U)$, such that $(I - U)(y) = x$ and also $(I - V)(y) = x$. This implies that $(I - V)(D(U)) = X$ or equivalently $D(U) = (I - V)^{-1}(X) = D(V)$ and finally $U = V$. $\square$

Remark 3.19. From Proposition 3.8, it follows that the semigroup given by Theorem 3.18, whose generator is U is unique.

Corollary 3.20. Let $S : [0,\infty) \to \mathcal{L}_0(X)$ be a bounded C_0-semigroup and let U be its generator. Then $S(t)(x) = \lim_{\lambda\to\infty} e^{tU(\lambda)}(x)$, for all $t \geq 0$, and $x \in X$, uniformly on bounded intervals.

Proof. From Theorem 3.17, it follows that U fulfills the following conditions:

1. $\overline{D(U)} = X$.
2. $(0, \infty) \subset \rho_0(U)$.
3. For each $\alpha \in A$, there exists $M(\alpha) > 0$, such that

$q_\alpha (\lambda R(\lambda, U))^n \leq M(\alpha)$, for all $\lambda > 0$ and $n \geq 1$. On the other hand, using Theorem 3.18, one can construct a bounded C_0-semigroup $T : [0, \infty) \to \mathcal{L}_0(X)$ whose generator is U and $T(t)(x) = \lim\limits_{\lambda \to \infty} e^{tU(\lambda)}(x)$, for all $t \geq 0$ and $x \in X$. Since T and S have the same generator from Proposition 3.8, it follows that $T = S$. □

Example 3.21. Let $X = C(\mathbb{R})$ be the Fréchet space defined in Example 2.4, and let $T : [0, \infty) \to L(X)$ be the mapping defined by $(T(t)(f))(x) = e^{-t} f(x)$, for all $x \in \mathbb{R}$ and $f \in X$. It is clear that T is a semigroup. For each $n \in N^*$, we have $p_n (T(t)(f)) \leq p_n ((f))$, for all $f \in X$ and $t \geq 0$. Let now $f \in X$, $n \in N^*$, $x \in [-n, n]$ and $t \geq 0$. Then $p_n (T(t)(f) - f) \leq \left(1 - e^{-t}\right) p_n(f)$ and hence $\lim\limits_{t \to 0} T(t)(f) = f$. Therefore T is a C_0- bounded semigroup .Let now $f \in X$, $n \in N^*$ and $t \geq 0$. We have: $p_n \left(\frac{T(t)(f) - f}{t} + f \right) \leq \frac{\left| e^{-t} + t - 1 \right|}{t} p_n(f)$. This means that $\lim\limits_{t \to 0} \frac{T(t)(f) - f}{t} = -f$, consequently the infinitesimal generator of T is $V : X \to X$, $V(f) = -f$. One can obviously see that $\rho(V) = \mathbb{R} \setminus \{-1\}$ and $R(\lambda, V)(f) = \frac{f}{\lambda + 1}$. Moreover, for $\lambda > -1$ we have:

$$R(\lambda, V)(f) = \int_0^\infty e^{-\lambda t} T(t)(f)\, dt.$$

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