

# A note on the generic initial ideal for complete intersections.

by

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## Abstract

We prove that the  $d$ -component of the generic initial ideal, with respect to the reverse lexicographic order, of an ideal generated by a regular sequence of homogeneous polynomials of degree  $d$  is revlex in a particular, but important, case. Using this property, we compute the generic initial ideal for several complete intersection with strong Lefschetz property.

**Keywords:** complete intersection, generic initial ideal, Lefschetz property.

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## Introduction.

Let  $K$  be an algebraically closed field of characteristic zero. Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ . Let  $n, d \geq 2$  be two integers. We consider  $I = (f_1, \dots, f_n) \subset S$  an ideal generated by a regular sequence  $f_1, \dots, f_n \in S$  of homogeneous polynomials of degree  $d$ . We say that  $A = S/I$  is a  $(n, d)$ -complete intersection. Let  $J = \text{Gin}(I)$  be the generic initial of  $I$ , with respect to the reverse lexicographical (revlex) order (see [5, §15.9], for details).

We say that a property  $(P)$  holds for a generic sequence of homogeneous polynomials  $f_1, f_2, \dots, f_n \in S$  of given degrees  $d_1, d_2, \dots, d_n$  if there exists a nonempty open Zariski subset  $U \subset S_{d_1} \times S_{d_2} \times \dots \times S_{d_n}$  such that for every  $n$ -tuple  $(f_1, f_2, \dots, f_n) \in U$  the property  $(P)$  holds. We say that a set of monomials  $M \subset S$  is a *revlex set* if, given a monomial  $u \in M$ , then any other monomial greater than  $u$  in revlex order is also in  $M$ .

For any nonnegative integer  $k$ , we denote by  $J_k$  the set of monomials from  $J$  of degree  $k$ . Conca and Sidman proved that  $J_d$  is revlex if  $f_1, \dots, f_n$  is a generic regular sequence, (see [4, Theorem 1.2]). In the first part of this paper, we prove that  $J_d$  is a revlex set in another case, namely, when  $f_i \in k[x_i, \dots, x_n]$ . It is likely to be true that  $J_d$  is revlex for any  $(n, d)$ -complete intersection, but we do not have the means to prove this assertion.

We say that a homogeneous polynomial  $f$  of degree  $s$  is *semiregular* for  $S/I$  if the maps  $(S/I)_t \xrightarrow{f} (S/I)_{t+s}$  are either injective, either surjective for all  $t \geq 0$ . We say that  $S/I$  has the *weak Lefschetz property* (WLP) if there exists a linear form  $\ell \in S$ , semiregular on  $S/I$ . In such case, we say that  $\ell$  is a weak Lefschetz element for  $S/I$ . A theorem of Harima-Migliore-Nagel-Watanabe (see [6]) states that  $S/I$  has (WLP) in the case  $n = 3$ . We say that  $S/I$  has the *strong Lefschetz property* (SLP) if there exists a linear form  $\ell \in S$  such that  $\ell^b$  is semiregular on  $S/I$  for all integer  $b \geq 1$ . In this case, we say that  $\ell$

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is a strong Lefschetz element for  $S/I$ . Harima-Watanabe [7] and later Herzog-Popescu [8], proved that  $S/I$  has (SLP) if  $f_i \in k[x_i, \dots, x_n]$ , for all  $1 \leq i \leq n$ .

In the second section of our paper, we compute the generic initial ideal for some particular cases of  $(n, d)$ -complete intersections:  $(n = 4, d = 2)$ ,  $(n = 5, d = 2)$  and  $(n = 4, d = 3)$ . In order to do this, we suppose in addition that  $S/I$  has (SLP). Note that this property holds for generic complete intersection (see [9]) and also in the case when  $f_i \in k[x_i, \dots, x_n]$ . It was conjectured that (SLP) holds for any standard complete intersection. A theorem of Wiebe [12] states that  $S/I$  has (WLP) (respectively (SLP)) if and only if  $x_n$  is a weak (respectively strong) Lefschetz element for  $S/J$ , where  $J = \text{Gin}(I)$ . As Example 1.9 show, the hypothesis  $\text{char}(K) = 0$  and  $f_1, \dots, f_n$  is a regular sequence are essentials.

## 1 Generic initial ideal for $(n, d)$ -complete intersections.

Let  $I = (f_1, \dots, f_n) \subset S = K[x_1, \dots, x_n]$  be an ideal generated by a regular sequence  $f_1, \dots, f_n \in S$  of homogeneous polynomials of degree  $d$ . Let  $J = \text{Gin}(I)$  be the generic initial ideal of  $I$ , with respect to the revlex order. It is well known that the Hilbert series of  $S/J$  is the same as the Hilbert series of  $S/I$  and moreover,  $H(S/J, t) = H(S/I, t) = (1 + t + \dots + t^{d-1})^n$ . More precisely, we have:

**Proposition 1.1.** 1.  $H(S/J, k) = \binom{k+n-1}{n-1}$ , for  $0 \leq k \leq d-1$ .

2.  $H(S/J, k) = \binom{k+n-1}{n-1} - n \binom{j+n-1}{n-1}$ , for  $d \leq k \leq \left\lfloor \frac{n(d-1)}{2} \right\rfloor$  and  $j = k - d$ .

3.  $H(S/J, k) = H(S/J, n(d-1) - k)$ , for  $k \geq \left\lceil \frac{n(d-1)}{2} \right\rceil$ .

*Proof.* Use induction on  $n$ . Denote  $H_n(t) = (1 + t + \dots + t^{d-1})^n$ . The case  $n = 1$  is trivial. The induction step follows from the equality  $H_n(t) = H_{n-1}(t)(1 + t + \dots + t^{d-1})$ .  $\square$

**Corollary 1.2.** 1.  $|J_k| = 0$ , for  $k \leq d-1$ .

2.  $|J_k| = n \binom{j+n-1}{n-1}$ , for  $d \leq k \leq \left\lfloor \frac{n(d-1)}{2} \right\rfloor$  and  $j = k - d$ .

3.  $|J_k| = \binom{\left\lceil \frac{n(d-1)}{2} \right\rceil + j + n - 1}{n-1} - \binom{\left\lfloor \frac{n(d-1)}{2} \right\rfloor - j + n - 1}{n-1} + n \binom{\left\lfloor \frac{n(d-1)}{2} \right\rfloor - d - j - n}{n-1}$ , for  $\left\lceil \frac{n(d-1)}{2} \right\rceil \leq k \leq (n-1)(d-1) - 1$ , where  $j = k - \left\lceil \frac{n(d-1)}{2} \right\rceil$

4.  $|J_k| = \binom{(n-1)d+j}{n-1} - \binom{n-1+d-1-j}{n-1}$ , for  $(n-1)(d-1) \leq k \leq n(d-1)$ , where  $j = k - (n-1)(d-1)$ .

*Proof.* Using  $|J_k| = |S_k| - H(S/J, k)$  the proof follows from 1.1.  $\square$

Suppose  $f_i = \sum_{k=1}^N b_{ik} u_k$  for  $1 \leq i \leq n$  where  $u_1, u_2, \dots, u_N \in S$  are all the monomials of degree  $d$  decreasing ordered in revlex and  $N = \binom{d+n-1}{n-1}$ . We denote  $u_k = x^{\alpha_k}$ . For example,  $\alpha_1 = (d, 0, \dots, 0)$ ,  $\alpha_2 = (d-1, 1, 0, \dots, 0)$  etc.

We take a generic transformation of coordinates  $x_i \mapsto \sum_{j=1}^n c_{ij} x_j$  for  $i = 1, \dots, n$ . Conca and Sidman proved in [4] that we may assume that  $c_{ij}$  are algebraically independents over  $K$ . More precisely, if we consider the field extension  $K \subset L = K(c_{ij} | i, j = \overline{1, n})$  and if we set

$$F_i = f_i \left( \sum_{j=1}^n c_{1j} x_j, \dots, \sum_{j=1}^n c_{nj} x_j \right) \in L[x_1, \dots, x_n], \quad i = 1, \dots, n$$

then  $J = \text{Gin}(I) = \text{in}(F_1, \dots, F_n) \cap S$ .

We write  $F_i = \sum_{j=1}^n a_{ij} u_j + \dots$  the monomial decomposition of  $F_i$  in  $L[x_1, \dots, x_n]$ . With these notations, we have the following elementary lemma:

**Lemma 1.3.**  *$J_d$  is revlex if and only if the following condition is fulfilled:*

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

*Proof.* Suppose  $\Delta \neq 0$ . Since  $|J_d| = n$ , it is enough to show that  $u_1, \dots, u_n \in J$ . Let  $A = (a_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, n}}$ . Since  $\Delta = \det(A) \neq 0$ ,  $A$  is invertible and we have

$$A^{-1} \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix},$$

where  $H_i = u_i +$  small terms in revlex order. Therefore  $LM(H_i) = u_i \in J$ , for all  $1 \leq i \leq n$ , where  $LM(H_i)$  denotes the leading monomial of  $H_i$  in the revlex order.

Conversely, since  $u_1, \dots, u_n \in J_d$ , we can find some polynomials  $H_i \in L[x_1, \dots, x_n]$ , with  $LM(H_i) = u_i$ ,  $1 \leq i \leq n$ , as linear combination of  $F_i$ 's. If we denote  $H_i = \sum_{j=1}^n \tilde{a}_{ij} u_j$  and  $\tilde{A} = (\tilde{a}_{ij})_{i,j=1, \dots, n}$ , it follows that there exists a map  $\psi : L^n \rightarrow L^n$ , given by a matrix  $E = (e_{ij})_{i,j=1, \dots, n}$ , such that  $\tilde{A} = A \cdot E$ . Now, since  $\det(\tilde{A}) \neq 0$  it follows that  $\Delta = \det(A) \neq 0$ , as required.  $\square$

**Remark 1.4.** By the changing of variables  $\varphi$  given by  $x_i \mapsto \sum_{j=1}^n c_{ij} x_j$ ,  $x^{\alpha_k}$  became

$$m_k := \left( \sum_{j=1}^n c_{1j} x_j \right)^{\alpha_{k1}} \cdots \left( \sum_{j=1}^n c_{nj} x_j \right)^{\alpha_{kn}} = \left( \sum_{|t|=\alpha_{k1}} c_1^t x^t \right) \cdots \left( \sum_{|t|=\alpha_{kn}} c_n^t x^t \right),$$

where, for any multiindex  $t = (t_1, \dots, t_n)$  we denoted  $x^t = x_1^{t_1} \cdots x_n^{t_n}$  and  $c_i^t = c_{i1}^{t_1} \cdots c_{in}^{t_n}$ . Let  $g_{kl}$  be the coefficient in  $c_{ij}$ 's of  $x^{\alpha_l}$  in the monomial decomposition of  $m_k$ . Using the above writing of  $m_k$ , we claim that:

$$(1) \quad g_{kl} = \sum_{\substack{|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn} \\ t_1 + \dots + t_n = \alpha_l}} \left[ \binom{\alpha_{k1}}{t_{11}} \cdots \binom{\alpha_{kn}}{t_{n1}} \right] \left[ \binom{\alpha_{k1} - t_{11}}{t_{12}} \cdots \binom{\alpha_{kn} - t_{n1}}{t_{n2}} \right] \cdots \\ \left[ \binom{\alpha_{k1} - t_{11} - \dots - t_{1n-1}}{t_{1n}} \cdots \binom{\alpha_{kn} - t_{n1} - \dots - t_{nn-1}}{t_{nn}} \right] \cdot c_1^{t_1} \cdots c_n^{t_n}.$$

Indeed, the monomial  $c_1^{t_1} \cdots c_n^{t_n}$  appear in the coefficient of  $x^\alpha$  in the expansion of  $m_k$  if and only if  $t_1 + \dots + t_n = \alpha_l$  and  $|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn}$ . Moreover, by Newton binomial, the coefficient of  $x_1^{t_{11}} \cdots x_n^{t_{1n}}$  in  $(\sum_{j=1}^n c_{ij} x_j)^{\alpha_{k1}}$  is  $\binom{\alpha_{k1}}{t_{11}} \binom{\alpha_{k1} - t_{11}}{t_{12}} \cdots \binom{\alpha_{k1} - t_{11} - \dots - t_{1n-1}}{t_{1n}} c_i^{t_i}$  for any  $1 \leq i \leq n$ , and thus we proved the claim.

Since  $a_{il} = \sum_{k=1}^N b_{ik} \cdot g_{kl}$ , from the Cauchy-Binet formula we get:

$$\Delta = \sum_{1 \leq k_1 < k_2 < \dots < k_n \leq N} B_{k_1, k_2, \dots, k_n} G_{k_1, k_2, \dots, k_n}, \text{ where}$$

$$B_{k_1, k_2, \dots, k_n} = \begin{vmatrix} b_{1k_1} & \cdots & b_{1k_n} \\ \vdots & & \vdots \\ b_{nk_1} & \cdots & b_{nk_n} \end{vmatrix} \text{ and } G_{k_1, k_2, \dots, k_n} = \begin{vmatrix} g_{k_1 1} & \cdots & g_{k_1 n} \\ \vdots & & \vdots \\ g_{k_n 1} & \cdots & g_{k_n n} \end{vmatrix}.$$

Now, we are able to prove the main result of our paper.

**Theorem 1.5.** *If  $f_i \in K[x_1, \dots, x_n]$  then  $J_d$  is revlex. In particular, if  $S/I$  is a monomial complete intersection, then  $J_d$  is revlex.*

*Proof.* Let  $k_i = \binom{i+d-1}{d}$ , for any  $i = 1, \dots, n$ . Then  $u_{k_i} = x_i^d$ . Recall our notation,  $u_k = x^{\alpha_k}$ . We have  $b_{11} \neq 0$ , otherwise  $I = (f_1, \dots, f_n) \subset (x_2, \dots, x_n)$  contradicting the fact that  $I$  is an Artinian ideal. Using a similar argument, we get  $b_{ik_i} \neq 0$  for all  $1 \leq i \leq n$ . Thus, multiplying each  $f_i$  with  $b_{ik_i}^{-1}$ , we may assume  $b_{ik_i} = 1$  for all  $1 \leq i \leq n$ . In other words,  $f_i = x_i^d + f'_i$ , where  $f'_i$  contains monomials smaller than  $x_i^d$  in the revlex order. Also, since  $f_i \in K[x_1, \dots, x_n]$  we have  $b_{i'k_i} = 0$  for any  $i' > i$ . In particular,  $B_{k_1, \dots, k_n} = 1$ .

In the expansion of the determinant  $G_{k_1, \dots, k_n}$ , appears the term  $g_{k_1 1} \cdot g_{k_2 2} \cdots g_{k_n n} = r \cdot (c_{11}^d)(c_{21}^{d-1} c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n})$ , where  $r$  is a nonzero (positive) integer. Indeed, by (1), we have  $g_{11} = c_{11}^d$ ,  $g_{k_2 2} = d c_{21}^{d-1} c_{22}$  and, in general,  $g_{k_i i} = \text{some binomial coefficient} \cdot c_i^{\alpha_i}$ . We claim that  $m = (c_{11}^d)(c_{21}^{d-1} c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n})$  doesn't appear again in the expansion of  $\Delta$ .

Since  $f_i \in k[x_1, \dots, x_n]$ , in the monomials in  $(c_{tl})$  of  $a_{ij}$  there are no  $c_{tl}$ 's with  $t < i$ . Also, all the monomials of  $f'_i$  contain variables  $x_t$  with  $t > i$ . Corresponding to them, in  $a_{ij}$ 's there are  $c_{tj}$ 's with  $t > i$ . Thus in  $a_{il}$  the only monomials in  $c_{i1}, \dots, c_{in}$  of degree  $d$  comes from  $\varphi(x_i^d) = (\sum_{j=1}^n c_{ij} x_j)^d$ , the other monomials being multiples of some  $c_{tl}$  with  $t > i$ . Consequently, in the expansion of  $\Delta$ , the monomials of the type  $c_1^{\beta_1} \cdots c_n^{\beta_n}$ , where  $\beta_1, \dots, \beta_n$  are multiindices with  $|\beta_1| = \dots = |\beta_n| = d$  comes only from  $\varphi(x_1^d), \dots, \varphi(x_n^d)$ .

On the other hand, for any  $1 \leq i \leq n$ ,  $c_i^{\alpha_i}$  is unique between the monomials in  $c_{tl}$ 's from  $\varphi(x_n^d)$ , because they are of the type  $c_i^\gamma$ , where  $\gamma$  is a multiindex with  $|\gamma| = d$ . From these facts, it follows that the monomial  $m$  is unique in the monomial expansion of  $\Delta$  and occurs there with a nonzero coefficient. Thus  $\Delta \neq 0$  and by applying Lemma 1.3 we complete the proof of the theorem.  $\square$

**Remark 1.6.** In the case  $n = 2$  and  $n = 3$ ,  $J_d$  is revlex for any  $(n, d)$ -complete intersection. Indeed, in the case  $n = 2$ ,  $J$  itself is revlex since it is strongly stable. In the case  $n = 3$ , since  $|J_d| = 3$  and  $J$  is strongly stable, it follows that either (a)  $J_d = (x_1^d, x_1^{d-1}x_2, x_1^{d-2}x_2^2)$ , either (b)  $J_d = (x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3)$ . But in the case (b), the map  $(S/J)_{d-1} \xrightarrow{x_3} (S/J)_d$  is not injective, because  $x_1^{d-1} \neq 0$  in  $(S/J)_{d-1}$  and  $x_1^{d-1}x_3 = 0$  in  $(S/J)_d$ . This is a contradiction with the fact that  $x_3$  is a weak Lefschetz element on  $S/J$  and therefore,  $J_d$  is revlex.

**Lemma 1.7.** (a)  $a_{i1} = f_i(c_{11}, \dots, c_{n1})$  for all  $1 \leq i \leq n$ .

(b) If  $1 \leq l \leq n$  is an integer then the sequence  $a_{1l}, a_{2l}, \dots, a_{nl}$  is regular as a sequence of polynomials in  $K[c_{ij} | 1 \leq i, j \leq n]$ .

*Proof.* Substituting  $x_j = 0$  for  $j \neq 1$  in  $F_i$  we get (a). In order to prove (b), firstly notice that  $a_{1l}, a_{2l}, \dots, a_{nl}$  is a regular sequence on  $K[c_{11}, c_{21}, \dots, c_{n1}]$ , since  $f_1, \dots, f_n$  is a regular sequence on  $K[x_1, \dots, x_n]$  and  $c_{11}, c_{21}, \dots, c_{n1}$  are algebraically independent over  $K$ .

Let  $1 \leq l \leq n$  be an integer. We claim that

$$(*) \frac{K[c_{ij} | 1 \leq i, j \leq n]}{(a_{1l}, \dots, a_{nl}, c_{i1} - c_{ij}, 1 \leq i \leq n, 2 \leq j \leq n)} \cong \frac{K[c_{11}, c_{21}, \dots, c_{n1}]}{(a_{1l}, a_{2l}, \dots, a_{nl})}.$$

Indeed, by (1), if we put  $c_{ij} = c_{i1}$  for all  $1 \leq i \leq n$  and  $2 \leq j \leq n$  in the expansion of  $g_{kl}$  we obtain  $r_l \cdot g_{k1}$ , where  $r_l$  is a strictly positive integer, which depends only on  $l$ , and therefore,  $a_{il}$  became  $r_l \cdot a_{i1}$ . From (\*) it follows that  $a_{1l}, \dots, a_{nl}, c_{i1} - c_{ij}$  for  $1 \leq i \leq n, 2 \leq j \leq n$  is a system of parameters for  $K[c_{ij} | 1 \leq i, j \leq n]$  and thus  $a_{1l}, \dots, a_{nl}$  is a regular sequence on  $K[c_{ij} | 1 \leq i, j \leq n]$ , so we proved (b).  $\square$

As we noticed in Remark 1.6, for  $n = 3$ , the conclusion of Theorem 1.5 holds for any regular sequence  $f_1, f_2, f_3$  of homogeneous polynomials of degree  $d$ . In the following, we give another proof of this, without using the fact that  $S/(f_1, f_2, f_3)$  has the (WLP), i.e.  $x_3$  is a weak Lefschetz element for  $S/J$ . Also, we get the same conclusion for the case  $n = 4$  and  $d = 2$ . However, this approach do not works in the general case.

**Proposition 1.8.** (a) If  $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$  is a regular sequence of homogeneous polynomials of degree  $d \geq 2$ ,  $I = (f_1, f_2, f_3)$  and  $J = \text{Gin}(I)$ , the generic initial ideal of  $I$ , with respect to the reverse lexicographical order, then  $J_d$  is a revlex set.

(b) If  $f_1, f_2, f_3, f_4 \in K[x_1, x_2, x_3, x_4]$  is a regular sequence of homogeneous polynomials of degree 2,  $I = (f_1, f_2, f_3, f_4)$  and  $J = \text{Gin}(I)$ , the generic initial ideal of  $I$ , with respect to the reverse lexicographical order, then  $J_2$  is a revlex set.

*Proof.* (a) Let  $A = (a_{ij})_{i,j=\overline{1,3}}$ . Since  $\text{Gin}(f_1, f_2)$  is strongly stable, it follows by Lemma 1.3 that  $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ . Analogously,  $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$  and  $\Delta_1 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$ . We have  $\Delta = a_{13}\Delta_1 - a_{23}\Delta_2 + a_{33}\Delta_3$ . Suppose  $\Delta = 0$ . It follows  $a_{13}\Delta_1 = a_{23}\Delta_2 - a_{33}\Delta_3$  and therefore, since  $a_{13}, a_{23}, a_{33}$  is a regular sequence in  $K[c_{ij} | i, j = \overline{1,3}]$ , we get  $\Delta_1 \in (a_{23}, a_{33})$ . The first three monomials of degree  $d$  in revlex order are  $x_1^d, x_1^{d-1}x_2$  and  $x_1^{d-2}x_2^2$ . It follows that the degree of  $a_{i1}, a_{i2}$  and  $a_{i3}$  in  $c_{21}, c_{22}, c_{23}$  is 0, 1, respectively 2, for any  $1 \leq i \leq 3$ .

Therefore, the degree of  $\Delta_1$  in the variables  $c_{21}, c_{22}, c_{23}$  is 1, but the degree of  $a_{23}$  and  $a_{33}$  in  $c_{21}, c_{22}, c_{23}$  is 2, which is impossible, since  $\Delta_1 \in (a_{23}, a_{33})$ .

(b) Let  $A = (a_{ij})_{i,j=\overline{1,4}}$ . Since any three polynomials from  $f_1, f_2, f_3, f_4$  form a regular sequence, it follows from (a) that any  $3 \times 3$  minor of the matrix  $\tilde{A} = (a_{ij})_{\substack{i=\overline{1,4} \\ j=\overline{1,3}}}$  is nonzero.

Let  $\Delta_i$  be the minor obtained from  $\tilde{A}$  by erasing the  $i$ -row. Suppose  $\Delta = 0$ . It follows that  $a_{14}\Delta_1 = a_{24}\Delta_2 - a_{34}\Delta_3 + a_{44}\Delta_4$  and therefore, since  $a_{14}, a_{24}, a_{34}, a_{44}$  is a regular sequence in  $K[c_{ij} | i, j = \overline{1,4}]$ , we get  $\Delta_1 \in (a_{24}, a_{34}, a_{44})$ . Since the first 4 monomials in revlex are  $x_1^2, x_1x_2, x_2^2, x_1x_3$ , we get a contradiction from the fact that the degree of  $\Delta_1$  in the variables  $c_{31}, c_{32}, c_{33}, c_{34}$  is zero, but the degree of  $a_{24}, a_{34}, a_{44}$  in  $c_{31}, c_{32}, c_{33}, c_{34}$  is 1.  $\square$

**Remark 1.9.** The hypothesis that  $K$  is a field with  $\text{char}(K) = 0$  is essential. Indeed, suppose  $\text{char}(K) = p$  and  $I = (x_1^p, x_2^p) \subset K[x_1, x_2]$ . Then, simply using the definition of the generic initial ideal, we get  $\text{Gin}(I) = I$  and, obviously,  $I_p = \{x_1^p, x_2^p\}$  is not revlex.

Also, the hypothesis that  $f_1, \dots, f_n$  is a regular sequence of homogeneous polynomials is essential. Let  $I = (f_1, f_2, f_3) \subset K[x_1, x_2, x_3]$ , where  $f_1 = x_1^2$ ,  $f_2 = x_1x_2$  and  $f_3 = x_1x_3$ . In order to compute the generic initial ideal of  $I$  we can take a generic transformation of coordinates with an upper triangular matrix, i.e.  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2 + c_{12}x_1$ ,  $x_3 \mapsto x_3 + c_{23}x_2 + c_{13}x_1$ , where  $c_{ij} \in K$  for all  $i, j$  (see [5, §15.9]). We get

$$F_1(x_1, x_2, x_3) := f_1(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = x_1^2,$$

$$F_2(x_1, x_2, x_3) := f_2(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{12}x_1^2 + x_1x_2,$$

$$F_3(x_1, x_2, x_3) := f_3(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{13}x_1^2 + c_{23}x_1x_2 + x_1x_3.$$

The generic initial ideal of  $I$ ,  $J = \text{in}(F_1, F_2, F_3)$  satisfies  $J_2 = I_2$ , but  $I_2$  is not revlex.

## 2 Several examples of computation of the Gin.

Let  $I = (f_1, \dots, f_n) \subset S = K[x_1, \dots, x_n]$  be an ideal generated by a regular sequence  $f_1, \dots, f_n \in S$  of homogeneous polynomials of degree  $d$ . Let  $J = \text{Gin}(I)$  be the generic initial ideal of  $I$ , with respect to the revlex order.

In [2], the case  $n = 3$  and  $d \geq 2$  is treated completely, when  $S/(f_1, f_2, f_3)$  has (SLP). More precisely, if  $d$  is odd, then

$$J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \text{ for } 1 \leq j \leq \frac{d-3}{2}, x_2^{\frac{3d-1}{2}}, x_3x_2^{\frac{3d-3}{2}},$$

$$x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \dots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}, 1 \leq j \leq \frac{d-3}{2}, x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d)$$

$$\text{or } J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \text{ for } 1 \leq j \leq \frac{d-4}{2}, x_1x_2^{\frac{3d-4}{2}}, x_2^{\frac{3d-2}{2}},$$

$$x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{3d-2}{2}-j}, 1 \leq j \leq \frac{d-2}{2}, x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d)$$

if  $d$  is even (see [2, Proposition 3.3]).

In the following, we discuss some particular cases with  $n \geq 4$ .

**The case  $n = 4, d = 2$ .** We assume that  $S/I$  has (SLP). From Wiebe's Theorem, it follows that  $x_4$  is a strong Lefschetz element for  $S/J$ . For a positive integer  $k$ , we denote  $Shad(J_k) = \{x_1, \dots, x_n\}J_k$ . We have  $H(S/J, t) = (1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$ .

We have  $|J_2| = 4$ . From Proposition 1.8,  $J_2$  is revlex, therefore

$$J_2 = \{x_1^2, x_1x_2, x_2^2, x_1x_3\} = \{\{x_1, x_2\}^2, x_1x_3\}.$$

We have  $|Shad(J_2)| = 12$ . On the other hand,  $|J_3| = 16$ , so we need to add 4 new generators at  $Shad(J_2)$  to get  $J_3$ . If we add a new monomial which is divisible by  $x_4^2$ , then the map  $(S/J)_1 \xrightarrow{\cdot x_4^2} (S/J)_3$ , will be no longer injective. Since  $|(S/J)_1| = |(S/J)_3|$ , we get a contradiction with the fact that  $x_4$  is a strong Lefschetz element for  $S/J$ . But there exists only 16 monomials in  $S$  which are not multiple of  $x_4^2$ . Thus

$$J_3 = \{\{x_1, x_2, x_3\}^3, x_4\{x_1, x_2, x_3\}^2\}, \text{ and therefore}$$

$$Shad(J_3) = \{\{x_1, x_2, x_3\}^4, x_4\{x_1, x_2, x_3\}^3, x_4^2\{x_1, x_2, x_3\}^2\}.$$

Since  $|Shad(J_3)| = 31$  and  $|J_4| = |S_4| - |(S/J)_4| = 35 - 1 = 34$  we have to add 3 new generators at  $Shad(J_3)$  in order to get  $J_4$ . Since  $J$  is strongly stable, these new generators are  $x_4^3x_1, x_4^3x_2$  and  $x_4^3x_3$ . So

$$J_4 = \{x_1, x_2, x_3, x_4\}^4 \setminus \{x_4^4\}. \text{ We get } Shad(J_4) = \{x_1, x_2, x_3, x_4\}^5 \setminus \{x_4^5\}$$

and since  $J_5 = S_5$  it follows that we must add  $x_4^5$  at  $Shad(J_4)$  to obtain  $J_5$ . From now on, we cannot add any new monomial.  $J$  is the ideal generated by all monomials added at some step  $k$  to  $Shad(J_k)$ , thus we proved the following proposition:

**Proposition 2.1.** *If  $I = (f_1, f_2, f_3, f_4)$  is an ideal generated by a regular sequence of homogeneous polynomials  $f_1, f_2, f_3, f_4 \in S = k[x_1, x_2, x_3, x_4]$  of degree 2 such that the algebra  $S/I$  has (SLP) then the generic initial ideal of  $I$  with respect to the revlex order is*

$$J = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3x_1, x_4^3x_2, x_4^3x_3, x_4^5).$$

*In particular, this assertion holds for a generic sequence of homogeneous polynomials  $f_1, f_2, f_3, f_4 \in S$  or if  $f_i \in k[x_i, \dots, x_4]$ ,  $1 \leq i \leq 4$ .*

**The case  $n = 5, d = 2$ .** In the following, we suppose that  $S/I$  has (SLP), so  $x_5$  is a strong Lefschetz element for  $S/J$ . Also, we suppose that  $J_2$  is revlex. We have  $H(S/J, t) = (1+t)^5 = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$ . We have  $|J_2| = 5$ . Since  $J_2$  is revlex from the assumption, we have  $J_2 = \{\{x_1, x_2\}^2, x_3\{x_1, x_2\}\}$ . So

$$Shad(J_2) = \{\{x_1, x_2\}^3, \{x_1, x_2\}^2\{x_3, x_4, x_5\}, x_3\{x_1, x_2\}\{x_3, x_4, x_5\}\}.$$

We have  $|Shad(J_2)| = 19$ . On the other hand  $|J_3| = |S_3| - |(S/J)_3| = 35 - 10 = 25$ , so we must add 6 new generators, from a list of 16 monomials, at  $Shad(J_2)$  to get  $J_3$ .

Since  $x_5$  is a strong Lefschetz element for  $S/J$  it follows that we cannot add any monomial of the form  $x_5 \cdot m$ , where  $m$  is nonzero in  $(S/J)_2$  because, in that case, the map  $(S/J)_2 \xrightarrow{x_5} (S/J)_3$  will be no longer injective. But there are  $|(S/J)_2| = 10$  such monomials  $m$ . Therefore, we must add the remaining 6 monomials,  $x_3^3, x_3^2x_4, x_1x_4^2, x_2x_4^2, x_3x_4^2, x_4^3$ . Thus

$$J_3 = \{\{x_1, x_2, x_3, x_4\}^3, x_5(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}. \text{ Therefore :}$$

$$\text{Shad}(J_3) = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}.$$

We have  $|\text{Shad}(J_3)| = 60$  and  $|J_4| = |S_4| - |(S/J)_4| = 70 - 5 = 65$ . So we need to add 5 new generators at  $\text{Shad}(J_3)$  to get  $J_4$ . If we add a monomial which is divisible by  $x_5^3$  we obtain a contradiction from the fact that the map  $(S/J)_1 \xrightarrow{x_5^3} (S/J)_4$  is no longer injective. Therefore, we must add:  $x_3^2x_5^2, x_1x_4x_5^2, x_2x_4x_5^2, x_3x_4x_5^2, x_4^2x_5^2$ , and so

$$J_4 = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2\{x_1, x_2, x_3, x_4\}^2\}.$$

$$\text{So } \text{Shad}(J_4) = \{\{x_1, x_2, x_3, x_4\}^5, \dots, x_5^3\{x_1, x_2, x_3, x_4\}^2\}.$$

We have  $|J_5| - |\text{Shad}(J_4)| = 4$ , so we must add 4 new generators at  $\text{Shad}(J_4)$  to get  $J_5$ . Since  $J$  is strongly stable, these new generators are:  $x_5^4x_1, x_5^4x_2, x_5^4x_3, x_5^4x_4$ . Therefore  $J_5 = \{\{x_1, \dots, x_5\}^5 \setminus \{x_5^5\}\}$ . Finally, we must add  $x_5^6$  to  $\text{Shad}(J_5)$  in order to obtain  $J_6$ . We proved the following proposition, with the help of [4, Theorem 1.2] and Theorem 1.5.

**Proposition 2.2.** *If  $I = (f_1, f_2, \dots, f_5) \subset K[x_1, \dots, x_5]$  is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 2 or if  $f_1, f_2, \dots, f_5$  is a regular sequence of homogeneous polynomials of degree 2 with  $f_i \in K[x_i, \dots, x_5]$  for  $i = 1, \dots, 5$  then  $J = \text{Gin}(I)$  the generic initial ideal of  $I$  with respect to the revlex order is:*

$$J = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^3, x_3^2x_4, x_1x_4^2, x_2x_4^2, x_3x_4^2, x_4^3, \\ x_3^2x_5^2, x_1x_4x_5^2, x_2x_4x_5^2, x_3x_4x_5^2, x_4^2x_5^2, x_5^4x_1, x_5^4x_2, x_5^4x_3, x_5^4x_4, x_5^6)$$

**The case  $n = 4, d = 3$ .** We suppose that  $S/I$  has (SLP), so  $x_4$  is a strong Lefschetz element for  $S/J$ . Also, we suppose that  $J_3$  is revlex.

$$\text{We have } H(S/J, t) = (1 + t + t^2)^4 = (1 + 2t + 3t^2 + 2t^3 + t^4)^2 =$$

$$= 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.$$

Since  $|J_3| = 4$  and  $J_3$  is revlex, it follows that  $J_3 = \{x_1, x_2\}^3$ . Therefore, we have  $\text{Shad}(J_3) = \{\{x_1, x_2\}^4, \{x_1, x_2\}^3\{x_3, x_4\}\}$ . Since  $|J_4| - |\text{Shad}(J_3)| = 4$ , we must add 4 new generators to  $\text{Shad}(J_3)$  to obtain  $J_4$ . Since  $x_4$  is a strong Lefschetz element for  $S/J$  we cannot add any monomial of the form  $x_4 \cdot m$ , where  $m \neq 0$  in  $J_3$ . Therefore, since  $J$  is strongly stable, we have to choose 3 monomials from the list  $x_3^2\{x_1, x_2\}^2, x_3^3\{x_1, x_2\}, x_3^4$ . There are two different chooses: either we add (I)  $x_3^2\{x_1, x_2\}^2$ , either (II)  $x_3^2x_1\{x_1, x_2, x_3\}$ .

In the case (I), we get  $J_4 = \{\{x_1, x_2\}^4, \{x_1, x_2\}^3\{x_3, x_4\}, x_3^2\{x_1, x_2\}^2\}$ , so

$$\text{Shad}(J_4) = \{\{x_1, x_2\}^5, \{x_1, x_2\}^4\{x_3, x_4\}, \{x_1, x_2\}^3\{x_3, x_4\}^2, x_3^2\{x_3, x_4\}\{x_1, x_2\}^2\}.$$

Since  $|J_5| - |Shad(J_4)| = 40 - 34 = 6$ , we need to add 6 new generators at  $Shad(J_4)$  to get  $J_5$ . Since  $x_4$  is a strong Lefschetz element for  $S/J$  we cannot add any monomial of the form  $x_4^2 m$ , where  $m$  is a nonzero monomial in  $J_3$ . So, we must add:  $x_3^4\{x_1, x_2, x_3\}, x_4 x_3^3\{x_1, x_2, x_3\}$ . Thus  $J_5 = \{\{x_1, x_2, x_3\}^5, x_4\{x_1, x_2, x_3\}^4, x_4^2\{x_1, x_2\}^3\}$ .

In the case (II), we have  $J_4 = \{\{x_1, x_2\}^4, \{x_1, x_2\}^3\{x_3, x_4\}, x_1 x_3^2\{x_1, x_2, x_3\}\}$ , so  $Shad(J_4)$  is the set  $\{\{x_1, x_2\}^5, \{x_1, x_2\}^4\{x_3, x_4\}, \{x_1, x_2\}^3\{x_3, x_4\}^2, x_3^2 x_1\{x_3, x_4\}\{x_1, x_2\}, x_3^3 x_1\{x_3, x_4\}\}$ . Since  $|J_5| - |Shad(J_4)| = 40 - 34 = 6$ , we must add 6 new generators at  $Shad(J_4)$  to get  $J_5$ . Since  $x_4$  is a strong-Lefschetz element for  $S/J$ , we cannot add any monomial of the form  $x_4^2 m$ , where  $m \neq 0$  in  $J_3$ . So, we must add:  $x_3^3 x_2^2, x_3^4 x_2, x_3^5, x_4 x_3^2 x_2^2, x_4 x_3^3 x_2, x_4 x_3^4$ . Thus

$$J_5 = \{\{x_1, x_2, x_3\}^5, x_4\{x_1, x_2, x_3\}^4, x_4^2\{x_1, x_2\}^3\},$$

the same as in the case (I). Thus, in both cases (I) and (II), we get:

$$Shad(J_5) = \{\{x_1, x_2, x_3\}^6, x_4\{x_1, x_2, x_3\}^5, x_4^2\{x_1, x_2, x_3\}^4, x_4^3\{x_1, x_2\}^3\}.$$

Since  $|Shad(J_5)| = |S_6| - 16$  and  $|J_6| = |S_6| - 10$ , we must add 6 new generators to  $Shad(J_5)$  in order to obtain  $J_6$ . Since  $x_4$  is a strong-Lefschetz element for  $S/J$ , these new generators are not divisible by  $x_4^4$ . So, we add  $x_4^3 x_3\{x_1, x_2\}^2, x_4^3 x_3^2\{x_1, x_2\}, x_4^3 x_3^3$  and thus,

$$J_6 = \{\{x_1, x_2, x_3\}^6, x_4\{x_1, x_2, x_3\}^5, x_4^2\{x_1, x_2, x_3\}^4, x_4^3\{x_1, x_2, x_3\}^3\}. \text{ So}$$

$$Shad(J_6) = \{\{x_1, x_2, x_3\}^7, x_4\{x_1, x_2, x_3\}^6, \dots, x_4^4\{x_1, x_2, x_3\}^3\}.$$

$|S_7| - |Shad(J_6)| = 6 + 4 = 10$  and  $|S_7| - |J_7| = 4$ , so we must add 6 new generators at  $Shad(J_6)$  to get  $J_7$ . Using the same argument, these new generators must be  $x_4^5\{x_1, x_2, x_3\}^2$  and therefore  $J_7 = \{\{x_1, x_2, x_3\}^7, x_4\{x_1, x_2, x_3\}^6, \dots, x_4^5\{x_1, x_2, x_3\}^2\}$ . We get

$$Shad(J_7) = \{\{x_1, x_2, x_3\}^8, x_4\{x_1, x_2, x_3\}^7, \dots, x_4^6\{x_1, x_2, x_3\}^2\}.$$

Since  $|S_8| - |Shad(J_7)| = 4$  and  $|S_8| - |J_8| = 1$ , we must add 3 new generators at  $Shad(J_7)$  in order to get  $J_8$ . Since  $x_4$  is strong-Lefschetz, these new generators are  $x_4^7\{x_1, x_2, x_3\}$ , so  $J_8 = \{x_1, x_2, x_3, x_4\}^8 \setminus \{x_4^8\}$ . Finally, we must add  $x_4^9$  to  $Shad(J_8)$  in order to obtain  $J_9$ . We proved the following proposition, with the help of [4, Theorem 1.2] and Theorem 1.5.

**Proposition 2.3.** *If  $I = (f_1, f_2, f_3, f_4) \subset K[x_1, x_2, x_3, x_4]$  is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 3 or if  $f_1, f_2, f_3, f_4$  is a regular sequence of homogeneous polynomials of degree 3 with  $f_i \in k[x_i, \dots, x_4]$ , for  $i = 1, \dots, 4$ , then  $J = Gin(I)$  the generic initial ideal of  $I$  with respect to the revlex order has one of the following forms:*

$$(I) \quad J = (\{x_1, x_2\}^3, x_3^2\{x_1, x_2\}^2, x_3^4\{x_1, x_2, x_3\}, x_4 x_3^3\{x_1, x_2, x_3\},$$

$$x_4^3 x_3\{x_1, x_2\}^2, x_4^3 x_3^2\{x_1, x_2\}, x_4^3 x_3^3, x_4^5\{x_1, x_2, x_3\}^2, x_4^7\{x_1, x_2, x_3\}, x_4^9)$$

$$(II) \quad J = (\{x_1, x_2\}^3, x_3^2 x_1\{x_1, x_2, x_3\}, x_3^3 x_2^2, x_3^4 x_2, x_3^5, x_4 x_3^2 x_2^2, x_4 x_3^3 x_2, x_4 x_3^4,$$

$$x_4^3 x_3\{x_1, x_2\}^2, x_4^3 x_3^2\{x_1, x_2\}, x_4^3 x_3^3, x_4^5\{x_1, x_2, x_3\}^2, x_4^7\{x_1, x_2, x_3\}, x_4^9)$$

**Remark 2.4.** *It seems Conca-Herzog-Hibi noticed in [3], page 838, that, if  $f_1, f_2, f_3, f_4$  is a generic sequence of homogeneous polynomials of degree 3 then the generic initial ideal  $J$  has the form (I), and  $J = Gin(x_1^3, x_2^3, x_3^3, x_4^3)$  has the form (II).*

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