

CIMPA-UNESCO-SMS School on Configuration
Spaces and Applications,
February 12-18, 2007,
School of Mathematical Sciences, GC University,
Lahore, Pakistan.

The program and the abstracts of the mini-courses

November 27, 2006

1 Program

- Monday 12th February: arrival day
- Tuesday 13th February
 - B1: Introduction to topology: π_1 and fibrations
 - B2: Introduction to topology: homology
 - K1: Introduction to topology: cohomology and duality
 - D1: Robotics, navigational complexity and algebraic topology
- Wednesday 14th February
 - D2: Robotics and algebraic geometry
 - J1: Configuration spaces
 - A1: Motion planning on configuration spaces
 - B3: Braid spaces of surfaces
- Thursday 15th February
 - A2: Non-classical configuration spaces
 - J2: Configuration spaces of R^n
 - K2: Braid spaces : properties and examples
 - B4: Configuration spaces of projective varieties
- Friday 16th February
 - A3: Computational algebraic topology
 - K3: Particle spaces and other analogs of braid spaces
 - Social Program in the afternoon
- Saturday 17th February
 - K4: Applications of particle spaces
 - A4: Wireless mobile networks
 - J3: Configuration spaces and cubical spaces
 - D3: Robotics and differential topology
- Sunday 18th February: departure day

2 B. Berceanu: Basic properties of configuration spaces

- **Lecture 1.** Introduction to topology: π_1 and fibrations
Fundamental groups and coverings, homotopy groups and fibrations; basic notions and examples.
- **Lecture 2.** Introduction to topology: homology.
CW complexes, Betti numbers, homology groups; basic notions and examples.
- **Lectures 3.** Braid spaces of surfaces.
Classical Artin braid group; compact surfaces, presentation of their braid groups.
- **Lecture 4.** Configuration spaces of projective manifolds.
Minimal models and formal spaces; compactification of configuration spaces and Fulton-MacPherson-Kriz model.

References

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- [4] A.Hatcher: *Algebraic topology*, Cambridge University Press (2003);
- [5] I.Kriz: *On the rational homotopy type of configuration spaces*, Ann. of Math. **139**, 227-237 (1994).

3 J.-Ph. Jourdan: An Introduction to Ordered Configuration Spaces of Points

- **Lectures 1. and 2.** Ordered configuration spaces : basic properties and first examples .

For manifolds M , we study the ordered configuration space $F(M, k)$ of all k pairwise distinct points in M , k a positive integer. Since Euclidean spaces form the building blocks of any manifold we will mainly focus on the case $M = \mathbf{R}^n$.

We introduce the Fadell-Neuwirth fibration which comes from projecting k -fold configurations to r -fold configurations with $r \leq k$.

We provide a braid-like description of some homotopy groups of $F(\mathbf{R}^n, k)$ and then give a complete description of the homology groups. Our formulation is phrased conveniently in terms of “planetary systems”.

We introduce cohomology groups, discuss the cup product through elementary examples and then describe the cohomology ring for configuration spaces of \mathbf{R}^n .

We finally construct explicit cellular models for $F(\mathbf{R}^n, k)$.

If time permits we will look into configuration spaces of spheres or other related examples.

- **Lecture 3.** Configuration spaces and cubical spaces

We discuss in some detail the configuration space of a smooth manifold M obtained by gluing together two manifolds A and $C \times [-1, 1]$. The point is to exhibit a homotopical cubical resolution of $F(M, k)$ defined from the configuration spaces of A and C . From this, we deduce a universal method for computing the pure braid group of a manifold. We illustrate the method in the case of the Mbius band. We seize the opportunity to discuss in this lecture some useful gluing constructions in homotopy theory and to introduce “the pure braid groups”.

References

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4 S. Kallel: An Introduction to Braid Spaces, Some Analogs and Applications

- **Lecture 1.** Cohomology and Duality.

We introduce cohomology and cup products. We discuss Poincaré and Alexander duality. This is a complement to lectures by Berceanu. This duality will be later applied to compute the homology and cohomology with mod-2 coefficients of braid spaces of the plane. This cute and elementary computation is due to D.B. Fuchs [4].

- **Lecture 2.** Basic Constructions and Examples

Braid Spaces are configuration spaces of unordered points on manifolds. As in the talks of Jourdan, the points making up a configuration are pairwise distinct but this time however, they cannot be distinguished. These so called braid spaces play a great role in algebraic topology [1] and in some parts of physics (as in gauge theory).

We introduce our spaces and describe the geometry and homotopy type of some basic examples like two fold and three fold configurations on spheres and euclidean space. We explain why these spaces are important in other parts of topology by relating them to loop spaces [5]. We will take the time to discuss new constructions, like fibre bundle structures, or new concepts, like orientability.

- **Lecture 3.** Particle Spaces and Analogs of Braid Spaces

We will discuss variants of braid spaces which come under the names of “truncated products”, “cyclic products” or “exponential spaces”. These differ mainly in the topology that is used to define them and which codifies the way the particles interact. The exponential spaces for instance are spaces of “finite subsets” and so particles (or configurations) can collide and merge. All these various constructions have useful applications that go beyond algebraic topology.

- **Lecture 4.** Applications and Further Structures

The fourth talk will either elaborate on or complement the above lectures depending on time and interest. Interesting further developments include defining operadic structures on braid spaces and using them for homology computations [2]. Another direction will be to look into billiard type problems [3].

References

- [1] G. Carlsson, R.J. Milgram, *Stable homotopy and iterated loop spaces*, Handbook of algebraic topology, Elsevier 1995.
- [2] F.R. Cohen, *The homology of C_{n+1} -spaces*, in Cohen-Lada-May, LNM 533 (1976).
- [3] M. Farber, S. Tabachnikov, *Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards*, Topology **41** (2002), no. 3, 553–589.
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5 M. Abubakr: Motion Planning and Computational Topology

- **Lecture 1.** Motion planning on configuration spaces: Topological complexity and trajectory design. (Farber, Koditschek, Yuzvinsky) [1, 2]

In this lecture, the topological description of *motion planning* on configuration spaces of mechanical systems will be explored. We will study Farber's concept of *topological complexity* [1] and its relationship to the design of practical motion planning algorithms. Finally, we will look at some standard engineering techniques that employ concepts from Morse theory and differential topology to design global navigation strategies for complicated configuration spaces arising in real-world mechanical systems [2].

- **Lecture 2.** Non-classical configuration spaces: Distributed robotics and reconfigurable systems. (Abrams, Ghrist) [3, 4]

In this lecture, we will study some non-classical configuration spaces such as the configuration spaces and braid groups of graphs [3]. We will see that the topology of these spaces are topologically interesting: they are $K(G, 1)$ and homologically bounded. We will also study the cubical complexes arising in the study of *reconfiguration*. The fact that these complexes have CAT(0) geometry leads to the design of unique optimal trajectories in these reconfigurable systems [4]. We will briefly review the application of these ideas to *metamorphic robotic systems* and *self assembly* in molecular biology.

- **Lecture 3.** Computational algebraic topology and its applications. (Carlsson, Herlihy, Mischaikow, Smale) [5]

In this part of the mini-course, we will briefly survey the emerging field of computational algebraic topology and its numerous applications in visualization, learning, dynamical systems, computer graphics and distributed algorithms [5]. For getting a flavor of the mathematics behind this area, we will present the *persistent homology* of filtrations of simplicial complexes. These simplicial complexes arise as either the Čech or the Rips-Veitoris complexes of point cloud data sets in some of the applications described above.

- **Lecture 4.** Wireless mobile networks meet algebraic topology : Homological methods for networked sensing and control (de Silva, Ghrist, Muhammad) [6, 7]

In this lecture, we will study the application of computational homological methods to the study of wireless sensor networks [6, 7]. We will see how to model radio coverage loss and network holes using Čech homology theory. We will also mention the use of discrete Hodge theory and Laplace-Beltrami operators for computing the homology groups by a distributed algorithm. Finally, we will give a cohomological interpretation to “flocking” and agreement behaviors in various social, biological and artificial networks. We will see that this interpretation gives us clues towards understanding the role of the topological structure of a network in its dynamical evolution.

References

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6 A. Dimca: Robotics and configuration spaces

- **Lecture 1.** Robotics, navigational complexity and algebraic topology

We show how the topological complexity of the configuration space of a robot, described in the language of algebraic topology, influences the complexity of the corresponding robot motion planning algorithms, see for details [2] and [7].

There is a particularly interesting special case of robot motion planning, where one may use the detailed knowledge of the topology of the subspace arrangements complements, see for details [3].

- **Lecture 2.** Robotics and algebraic geometry.

We show how even simple robots lead to difficult questions in algebraic geometry, and how the powerful tool of algebraic geometry may be used to answer such questions, see for details [1], [4], [6].

- **Lecture 3.** Robotics and differential topology.

We show that the methods of differential topology, e.g. Morse theory, can play a key role in understanding the configuration spaces of some simple robots, see for details [1], [5].

References

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